Topics on the Boundedness of Fourier Multipliers over Group Algebras

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Contents

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Muchas decisiones importantes en nuestra vida son tomadas sin conocimiento fiable sobre como saldrán. Yo ya sabía, allá por 4 de carrera, que me quería dedicar a la investigación. No obstante, la decisión de hacerlo en una rama concreta y bajo una supervisión concreta si fue tomada *sin conocimiento*. En el verano de 2010 elegí asistir a la escuela JAE y hacer un trabajo de iniciación a la investigación con Javier Parcet sobre "La Transformada de Fourier en Grupos Topológicos". Mi inclinación natural en aquel momento era hacia la geometría. De hecho, de todos los matemáticos a los que escribí Javi era el único analista, también fue de los pocos que me respondió el correo. Justo 6 años después puedo decir que la decisión, aleatoria, de trabajar con él fue acertada. En los 4 años que he pasado haciendo este doctorado me he beneficiado, no solo del talento y la capacidad de trabajo de Javi, si no también de un clima de respeto y libertad intelectual entre sus estudiantes que, con la perspectiva que me han dado estos años, puedo decir que es anómalo en el mundo académico.

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Contexto y Resultados Principales

El principio que afirma que algunos espacios singulares pueden ser entendidos mediante el estudio de las álgebras no conmutativas que describen los observables sobre dicho espacio es una de las piedras angulares de lo que se ha venido en llamar geometría no conmutativa. Uno de dichos espacios singulares que admite una descripción natural en términos de álgebras no conmutativas es el dual de un grupo topológico. Recordemos que si G es un grupo abeliano y localmente compacto podemos definir su dual $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$ como el grupo de todos los caracteres continuos $\chi : G \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ con la multiplicación puntual. En este contexto, el teorema de dualidad de Pontriaguin afirma que $G^{\wedge\wedge} = G$. Dicha relación falla cuando G no es abeliano[1]. Sin embargo, podemos definir tanto álgebras generalizando $L_\infty(\hat{G})$, en caso de querer estudiar la teoría de la medida no conmutativa, como álgebras generalizando $C_0(\hat{G})$ en caso de querer estudiar la topología no conmutativa del dual de G. Denotaremos por $\mathcal{L}G$ la primera álgebra, construida como una clausura del álgebra de grupo $\mathbb{C}[G]$. $\mathcal{L}G$ posee una aplicación positiva y σ-aditiva $\tau : \mathcal{L}G_+ \to [0, \infty]$, conocida como el peso de Plancherel, que juega el mismo papel que la integral contra la medida de Haar de $\hat{G}$. Dado un peso en un álgebra de von Neumann hay una teoría bien desarrollada de integración no conmutativa que permite definir los espacios L_p no conmutativos $L_p(\mathcal{L}G)$ como complecciones de los operadores $x \in \mathcal{L}G$ tales que[2]

$$\tau(|x|^p) < \infty.$$

El problema que vamos a considerar en esta tesis consiste en determinar cuando un multiplicador de Fourier T_m está acotado o completamente acotado en $L_p(\mathcal{L}G)$. Recordemos que el multiplicador de Fourier de un símbolo $m \in L_\infty(G)$ es el operador (posiblemente no acotado) dado por extensión de $g \mapsto m(g)\, g$ definido sobre $\mathbb{C}[G]$. Al igual que en el caso abeliano, no es posible dar un criterio necesario y suficiente en términos de m para determinar cuando T_m esta acotado en $L_p(\mathcal{L}G)$, excepto cuando p está el rango $p \in \{1, 2, \infty\}$. Todo lo que podemos aspirar a obtener son, o bien condiciones necesarias, o bien suficientes. El estudio de la acotación completa de dichos operadores recibió un impulso pionero con los trabajos de Haagerup [Haa78] y Cowling-Haagerup [CH89] en relación con la amenabilidad débil de ciertos grupos. Tambien fue estudiado por Pisier [Pis95a] en relación con ciertas series lacunares. Sin embargo, el caso de p general ha permanecido prácticamente sin explorar hasta décadas más recientes. Entre la investigación sobre el problema destacamos los trabajos de Harcharras [Har99] en conexión con los multiplicadores de Schur y los trabajos de Lafforgue-de la Salle [LdlS11] en conexión con las propiedades de aproximación de ciertos retículos $\Gamma \leq \mathrm{SL}_r(F)$. También cabe destacar el trabajo de Bożejko y Fendler [BF06] que usa un enfoque basado en transformadas de polinomios ortogonales. Un avance reciente en la teoría L_p se inaugura con [JMP14a]. Paralelamente al desarollo de esta tesis, Junge, Mei y Parcet han obtenido cotas adimensionales para las transformadas de Riesz generalizadas [JMP14c] y Parcet y Rogers han estudiado transformadas de Hilbert en ciertas extensiones de $\mathbb{R}^n$ [PR16], asimismo Caspers, Parcet, Perrin y Ricard han obtenido teoremas de estabilidad algebraica paralelos a los

[1] Por ejemplo tomemos G simple, en tal caso $\mathrm{Hom}(G, \mathbb{T}) = \{e\}$

[2] Esto no es totalmente correcto ya que la expresión $x \mapsto \tau(|x|^p)^{1/p}$ solo es una norma si τ es tracial. El peso de Plancherel solo es una traza si G es unimodular. En otro caso, para definir los espacios L_p es necesario usar o bien interpolación compleja o bien teoría espacial de álgebras de von Neumann.

debidos a de Leuuw [CPPR15]. No obstante, más alla de las contribuciones aquí listadas, el campo continúa fundamentalmente inexplorado.

Multiplicadores de Fourier suaves

Nuestro primer objetivo es obtener teoremas de multiplicadores suaves, esto es resultdos que afirman que si m tiene una cantidad finita de derivadas adecuadamente localizadas entonces T_m está acotado en L_p.

Dicho problema tiene interés por dos motivos poderosos. El primero es que el número de derivadas requeridas es una cantidad dependiente de la dimensión y por tanto, si queremos extender la teoría al caso no conmutativo, es necesario dirimir cual es la dimensión de $\mathcal{L}G$ como espacio no conmutativo. Dicho problema parece íntimamente relacionado con la teoría de espacios métricos no conmutativos. La definición de dichos espacios ha atraido bastante atención en el pasado tanto desde el campo de las C^*-álgebras, veanse los trabajos de Rieffel [Rie04b, Rie02, Rie04a], como desde el campo de las W^*-álgebras, veanse los trabajos de Kuperberg y Weaver [KW12, Wea12]. La segunda motivación es que, a menudo, condiciones naturales en términos del análisis armónico de $\mathcal{L}G$ se traducen en propiedades estudiadas en teoría geométrica de grupos, creandose así un puente entre dos áreas aparentemente disjuntas de la matemática.

El primer resultado que nos gustaría generalizar al contexto no conmutativo es el teorema espectral de Hörmander-Míjlin. Como la mayoría de resultados cuantitativos del análisis armónico, dicho teorema no admite una formulación única que abarque todos los posibles casos y debe ser entendido más bien como un patrón que pasamos a describir. Sea (X, μ) un espacio de medida y $S_t = e^{-t\,A} : L_2(X) \to L_2(X)$ un semigrupo markoviano. Dicho semigrupo codifica una métrica natural d_Γ, conocida como distancia gradiente, vease [SC09] para una definición precisa. Asumamos que

(i) A is un operador local (i.e. un operador que preserva los soportes).

(ii) A satisface alguna desigualdad de tipo Sobolev con dimensión D.

(iii) (X, μ, d_Γ) es doblante como espacio métrico de medida, i.e. $\mu(B_x(2\,r)) \leq C\,\mu(B_x(r))$.

En tal caso, para todo $s > D/2$, se tiene que

$$\big\| m(A) : L_p(X) \to L_p(X) \big\| \quad \lesssim_{(p)} \quad \sup_{t \geq 0} \| m(t\,\cdot)\,\eta \|_{W^{\infty,s}(\mathbb{R}_+)} \qquad \text{para todo } 1 < p < \infty,$$

donde $\eta \in C_c^\infty(\mathbb{R}_+)$ y $W^{\infty,s}(\mathbb{R}_+)$ es el espacio de Sobolev fraccionario dado por $\|f\|_{W^{\infty,s}(\mathbb{R}_+)} = \|(\mathbf{1} + A)^{s/2} f\|_\infty$.

El principal ejemplo que el lector debe tener en mente es el de una variedad riemaniana M con $A = -\Delta$ el operador de Laplace-Beltrami. En ese caso d_Γ es la distancia usual y las desigualdades de Sobolev imponen restricciones geométricas en M, típicamente en el crecimiento de sus bolas. El esquema de la prueba es como sigue

1. Usar la localidad de A y las desigualdades de Sobolev para obtener decaimiento lejos de la diagonal del núcleo $k_t(x,y)$ de S_t, típicamente cotas gausianas.

2. Expresar $m(A)$ como una combinación convexa de elementos de la forma S_z, para $0 \leq \Re\{z\}$ y usar análisis complejo para extender las cotas gausianas de $\mathbb{R}_+$ a todo el semiplano complejo.

3. Usar las cotas gaussianas de S_z para obtener cierto decaimiento lejos de la diagonal de $m(A)$. Ese decaimiento implica que $m(A)$ es un operador de Calderón-Zygund. Finalizamos usando teoría de Calderón-Zygmund para espacios doblantes.

El paso 1 puede ser llevado a cabo de multiplies maneras. Por ejemplo en [Sik96, Sik04] la velocidad de propagación finita de la ecuación de ondas es usada para obtener cotas gausianas a partir de ciertas desigualdades de Sobolev, principalmente ultracontractividad. Otra enfoque usando iteración de Mosser puede encontrarse en [Cou93, DP89, Dav90]. También seañalamos [SC02] para la misma implicación usando desigualdades de Sobolev invariantes por reescalamieto en conjunción con desigualdades de Harnack parabólicas. Como ya hemos indicado antes, el esquema de arriba da resultados en distintos contextos. Históricamente, resultados en dicha dirección fueron obtenidos en [Chr91, HS] para grupos de Lie nilpotentes, en [Ale01] y [Ale94] para multiplicadores de Hörmander-Mikhlin en grupos discretos y grupos de Lie de crecimiento polinómico y en [Heb95] para espacios métricos de medida generales.

Para generalizar el esquema de arriba en el contexto no conmutativo necesitamos un sustituto de los semigroups markovianos. Tomaremos como tal los semigrupos de multiplicadores de Fourier unitales y completamente positivos $\lambda_g \mapsto e^{t\,\psi(g)}\lambda_g$, vease la section 1.6. Como veremos más adelante, un semigrupo de dicha forma es Markoviano sii $\psi : G \to \mathbb{R}_+$ es una función de longitud condicionalmente negativa. Tambien es claro que $m(A) = T_{m\circ\psi}$. Enunciar desigualdades de Sobolev en este contexto es relativamente fácil. El problema viene de que ni el paso 1 ni el paso 3 tienen un análogo claro en el caso no conmutativo. Para salvar la dificultad en el paso 1 impondremos cotas gaussianas desde el principio a nuestro semigrupo. Por otro lado, aunque recientemente se han dado pasos para la formulación de una teoría de Calderón-Zygmund no conmutativa, ver [Par09, GPJP16, JMP14b], esta aún carace de la madurez necesaria para ser usada en este contexto. Para superar dicha dificultad usaremos un principio de acotación de multiplicadores por operadores maximales. Los operadores que nos aparecerán serán comparables a operadores de Hardy-Littlewood. En el caso clásico. la acotación de dichos operadores se sigue de la propiedad doblante. No obstante, un resultado en esas lineas no parece ser cierto en el caso no conmutativo. Por tanto tenemos que imponer adicionalmente que nuestro espacio tenga un maximal de Hardy-Littlewood acotado. Cuando tenemos cotas gausianas, una medida doblante y acotación para el maximal de Hardy-Littlewood diremos que nuestro espacio tiene las hipótesis estándar. Para espacios con las hipótesis estandar probamos un teorema de Hörmander-Míjlin para órdenes de suavidad $s > D/2$, donde D es la dimension de Sobolev. Como aplicación obtenemos un teorema similar al teorema principal de [JMP14a].

En el capítulo 1 daremos un breve repaso a las herramientas que utilizaremos en este texto. En el capítulo 2 exponemos el principio de acotación por operadores maximales. Como aplicación obtenemos un teorema similar al teorema de Marcinkiewicz en grupos de Lie de crecimiento polinómico, que exponemos en la Sección 2.2. El teorema espectral de Hörmander-Míjlin se prueba en el capítulo 3.

Transferencia para productos cruzados

Los principales resultados de este bloque estarán contenidos en el capítulo 4. Sea G un grupo y $\theta : G \to \mathrm{Aut}(\mathcal{M})$ una acción que preserva la traza $\tau : \mathcal{M}_+ \to [0,\infty]$. Sabemos que en ese caso podemos levantar la traza τ a un peso en el producto cruzado $\mathcal{M} \rtimes_\theta G$ de manera canónica y que dicho peso es tracial sii G es unimodular. Esto da pie a definir espacios L_p sobre el producto cruzado $L_p(\mathcal{M} \rtimes_\theta G)$ y estudiar sus propiedades. Nuestro objetivo aquí será generalizar resultados anteriores de Neuwirth-Ricard [NR11] y Caspers-de la Salle [CdlS15] que afirman que, para grupos amenables, la acotación completa del multiplicador de Herz-Schur $M_m : S_p(L_2G) \to S_p(L_2G)$ dado por

$$M_m([a_{g\,h}]) = [m(g\,h^{-1})\,a_{g\,h}],$$

implica la acotación completa del multiplicador de Fourier $T_m : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)$, donde $S_p(L_2G)$ son las así llamadas clases de Schatten, dadas por $S_p(H) = L_p(\mathcal{B}(H), \mathrm{Tr})$. Su prueba usa la amenabilidad de G para obtener una sucesión $j_{p,\alpha} : L_p(\mathcal{L}G) \to S_p(L_2(G))$ asintóticamente isométrica y técnicas basadas en ultraproductos para pasar al límite. Nosotros generalizaremos sus resultados de los grupos amenables a las acciones amenables. Concretamente, si θ es amenable obtenemos una isometría completa

$$L_p(\mathcal{M} \rtimes_\theta G) \xrightarrow{\;j_p\;} \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2G))$$

que conjuga $\mathrm{Id} \rtimes T_m$ y $\mathrm{Id} \otimes M_m$. Puesto que es bien conocido que la acotación completa de T_m implica la acotación completa de M_m obtenemos que

$$\|\mathrm{Id} \rtimes T_m\|_{\mathrm{cb}} \leq \|\mathrm{Id} \otimes M_m\|_{\mathrm{cb}} \leq \|T_m\|_{\mathrm{cb}}.$$

Una ventaja de nuestro método es que, además de permitirnos acotar multiplicadores de Fourier, permite obtener cotas para extensiones de la forma $S \rtimes \mathrm{Id}$, donde S es un operador θ-equivariante. En particular, asumiendo que la secuencia aproximante de θ satisface cierta condición de acretividad, tenemos que

$$\|S \rtimes \mathrm{Id}\|_{\mathrm{cb}} \leq C^{\frac{1}{p}}\,\|S\|_{\mathrm{cb}},$$

donde $1 \leq C$ es una constante midiendo la acretividad. La primera aplicación de nuestros resultados es una prueba de la estabilidad bajo productos cruzados de operadores maximales completamente positivos. Como corolario obtenemos la estabilidad de las hipóesis estandard que definimos en el capítulo 3. Nótese que, puesto que nuestro teorema no requiere que el grupo G sea amenable, puede tener aplicaciones potenciales para acotar multiplicadores sobre grupos exactos más generales. Esta y otras cuestiones son discutidas en la sección 4.4 del capítulo 4.

Ideales en $\mathcal{M} \otimes_{eh} \mathcal{M}$ y métricas W^*

Ya hemos visto como en el caso clásico, las cotas gausianas pueden obtenerse de la conjunción de desigualdades de tipo Sobolev —en particular ultracontractividad— y propiedades geométricas, ya sea velocidad de propagación finita o localidad del generador. Los resultados que vamos a describir aquí pueden entenderse como un primer paso hacia la obtención de dicho teorema en el contexto no conmutativo. Para ello estudiaremos una clase de métricas no conmutativas, introducidas recientemente por Kuperberg y Weaver [Wea12, KW12], conocidas como métricas W^*. Dichas métricas parecen especialmente bien adaptadas a la hora de describir la velocidad de propagación finita o la localidad. El concepto de métrica W^* se define sobre el de relación cuántica. Una relacion cuántica sobre $\mathcal{M} \subset \mathcal{B}(H)$ es un $\mathcal{M}'$-bimódulo cerrado en la topología débil-$*$. La intuición aquí viene de que, en el caso de $\mathcal{M} = \ell_\infty(X)$, dichos bimódulos están conpuestos por los operadores $[a_{x\,y}]_{x,y \in X} \in \mathcal{B}(\ell_2 X)$ cuya matríz $a_{x\,y}$ está soportada en un conjunto fijado $R \subset X \times X$ y de ahí la conexión con las relaciones clásicas. Una métrica W^* no es más que una familia de relaciones cuánticas comportandose como las relaciones $\{(x,y) : d(x,y) \leq r\}$, esto es, satisfaciendo propiedades análogas a la desigualdad triangular, la simetría, etc.

Entre otras caracterizaciones, Weaver demostró que cuando $\mathcal{M}$ es finito dimensional, las relaciones cuánticas sobre $\mathcal{M}$ están en biyección con proyecciones sobre $\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}$. Nosotros generalizamos ese resultado para álgebras generales cambiando el producto tensorial espacial por el producto extendido de Haagerup $\mathcal{M} \otimes_{eh} \mathcal{M}$ y las proyecciones por ideales a la izquierda cerrados en la topología débil-$*$, véase [Smi91, BS92a] para la construcción del producto extendido de Haagerup. El motivo por el que usamos ese producto tensorial es que es isomorfo al álgebra de operadores completamente acotados y $\mathcal{M}'$-bimodulares a través de la aplicación $\Phi : \mathcal{M} \otimes_{eh} \mathcal{M} \to \mathcal{CB}^{\sigma}_{\mathcal{M}',\mathcal{M}'}(\mathcal{B}(H))$ dada por extensión de

$$\Phi_{x \otimes y}(T) = x\, T\, y.$$

La biyección entre realaciones cuánticas e ideales no es más que una relación de doble aniquilador entre $\mathcal{M}'$-bimódulos y operadores $\mathcal{M}'$-bimodulares. En el caso de relaciones cuánticas sobre $\mathcal{L}G$ invariantes por la comultiplicación natural tenemos que podemos pasar de un ideal $J \subset \mathcal{M} \otimes_{eh} \mathcal{M}$ a un ideal $J \subset MG$, donde MG es el álgebra de medidas de Borel. Este proceso es una manera no conmutativa de eliminar una variable pasando de trabajar con objetos invariantes en $G \times G$ a trabajar con objetos en G. Al final del capítulo, en la sección 5.5, explicamos cómo usar los ideales $J \subset \mathcal{M} \otimes_{eh} \mathcal{M}$ para definir el decaimiento lejos de la diagonal de un operador.

Overview and Main Results

The fact that certain singular objects can be better understood by studying noncommutative algebras generalizing the natural functions over that object is a fruitful principle that constitutes the main intuition of what is now known as noncommutative geometry. One of the singular objects that admits a natural description in terms of noncommutative algebras is the dual of a group. Recall that if G is abelian and locally compact we can define its dual group $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$ as the group of continuous characters $\chi : G \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The Pontryagin duality theorems asserts that $G^{\wedge\wedge} = G$. Of course, such relation fails when G is nonabelian[3]. Nevertheless, we can define an algebra generalizing either $L_\infty(\hat{G})$, is we want to study the noncommutative measure theory of $\hat{G}$ or $C_0(\hat{G})$ is we want to study the noncommutative topology of $\hat{G}$. Since our interests here are more analytical than topological we shall center our study in the first algebra. Such algebra will be denoted by $\mathcal{L}G$ and by construction it is just a completion of the group algebra $\mathbb{C}[G]$. Over $\mathcal{L}G$ there is a positive and σ-additive map $\tau : \mathcal{L}G_+ \to [0, \infty]$, called the Plancherel weight, playing the same role as the Haar integral over $\hat{G}$. Given any such weight there is a well-developed noncommutative integration theory that allow us to define the noncommutative L_p-spaces $L_p(\mathcal{L}G)$ as completion of the operators $x \in \mathcal{L}G$ such that[4]

$$\tau(|x|^p) < \infty.$$

The problem we will be concerned with in this Ph.D. thesis is determining the boundedness or complete boundedness of Fourier multipliers T_m on $L_p(\mathcal{L}G)$. Recall that the Fourier multiplier of symbol $m \in L_\infty(G)$ is the (possibly unbounded) map extending the function $g \mapsto m(g)\, g$ defined over $\mathbb{C}[G]$. Pretty much like in the case of abelian groups there is no hope for a closed criterion for the boundedness of T_m in terms of m, when p is outside the range $p \in \{1, 2, \infty\}$, and all we can fight for are either necessary of sufficient conditions. The completely bounded theory of such multipliers in the case $p = \infty$ was pioneered by Haagerup [Haa78] and Haagerup-Cowling [CH89] in relation with weak amenability of groups and by Pisier [Pis95a] in relation to lacunary series. The general L_p case has remained quite unexplored until very recently. Among the recent research we highlight the works of Harcharras [Har99] in connection with Schur multipliers and of Lafforgue and de la Salle [LdlS11], who studied the problem for its connection with approximation properties of certain lattices in $\Gamma \leq \mathrm{SL}_r(F)$. We also mention the works of Bożejko and Fendler who used an approach based in orthogonal polynomial transforms [BF06]. A recent breakthrough in the L_p theory of multipliers appeared in [JMP14a]. During the time the research in this Ph.D. thesis have been conducted Junge, Mei and Parcet have developed dimension-free bounds for Riesz transforms [JMP14a], Parcet and Rogers have studied Hilbert transforms in certain extensions of $\mathbb{R}^n$ [PR16] and Caspers, Parcet, Perrin and Ricard have obtained algebraic stability results parallels to those of de Leuuw [CPPR15]. Besides those results, the area remains largely unexplored.

[3]for instance if G is simple, then $\mathrm{Hom}(G, \mathbb{T}) = \{e\}$

[4]This is not entirely correct. Indeed, the formula above only defines a norm when τ is a trace and the Plancherel weight is tracial iff G is unimodular. In other cases the L_p spaces are defined either using spatial theory of von Neumann algebras or through interpolation

Smooth Fourier multipliers

Our first goal will be to obtain smooth Fourier multipliers theorems, i.e. results stating that if m has a finite number of, suitable localized, derivatives, then T_m is completely bounded.

We find interesting such problems mainly for two reasons. The first is that —since the optimal number of derivatives involved is, in the abelian case, a dimension-dependent quantity— generalizing such results in the noncommutative setting requires a description of the dimension of $\mathcal{L}G$, seeing as a noncommutative metric space. Studying noncommutative metric spaces is a notably difficult problem that have received much attention in the past, both from the C^*-algebra framework, see the work of Rieffel [Rie04b, Rie02, Rie04a], and from the W^*-algebra one, see the works of Kuperberg and Weaver [KW12, Wea12]. Other motivation for studying the problem is that it is often the case that natural conditions in terms of the noncommutative harmonic analysis of $\mathcal{L}G$ can be translated into properties of G studied in geometric group theory, creating a connection between two separate branches of mathematics.

The first result that we will like to generalize is the spectral Hörmander-Miklin multiplier theorem. Like most results in analysis, the theorem do not have an all-encompassing formulation, being true in different settings that sometimes don not admit a common description. We are going to describe here the main template that such result follows in the classical case. Let (X, μ) be a measure space endowed with a Markovian semigroup $S_t = e^{-t\,A} : L_2(X) \to L_2(X)$. Such semigroup encodes a natural metric d_Γ, called the gradient metric, see [SC09] for precise definitions. Assume that

(i) A is a local operator (i.e. an operator preserving supports).

(ii) A satisfies some Sobolev-type inequality with dimension D.

(iii) (X, μ, d_Γ) is a doubling metric measure space, i.e. $\mu(B_x(2\,r)) \leq C\,\mu(B_x(r))$.

Then, for every $s > D/2$, we have that

$$\big\|m(A) : L_p(X) \to L_p(X)\big\| \lesssim_{(p)} \sup_{t \geq 0} \|m(t\,\cdot)\,\eta\|_{W^{\infty,s}(\mathbb{R}_+)} \quad \text{for } 1 < p < \infty,$$

where $\eta \in C_c^\infty(\mathbb{R}_+)$ and $W^{\infty,s}(\mathbb{R}_+)$ is the fractional Sobolev space given by $\|f\|_{W^{\infty,s}(\mathbb{R}_+)} = \|(\mathbf{1} + A)^{s/2} f\|_\infty$.

The main example that the reader such keep in mind is that of a Riemannian manifold M with $A = -\Delta$ being its Laplace-Beltrami operator. In that case d_Γ is the usual distance and the Sobolev inequalities generally impose certain geometrical restrictions on M. The schema of the proof goes roughly as follows.

1. Use the locality of A and the Sobolev inequalities to get off-diagonal bounds, typically Gaussian bounds, for the kernel $k_t(x, y)$ of the semigroup S_t.

2. Express $m(A)$ as a convex combination of elements of the form S_z, for $0 \leq \Re\{z\}$. Complex analysis allows us to extend the Gaussian bounds from $\mathbb{R}_+$ to the complex half-plane.

3. Use the off-diagonal bounds of S_z to get off-diagonal bounds for the kernel $m(A)$ implying that $m(A)$ is a Calderón-Zygund operator. Then we can use Calderón-Zygmund Theory for doubling spaces.

The step 1 can be performed in several ways. For example in [Sik96, Sik04] finite speed of propagation is used to upgrade certain Sobolev inequalities, mainly ultracontractivity, to Gaussian bounds. Other approaches using Mosser iteration can be found in [Cou93, DP89, Dav90]. We also mention [SC02] for the implication using the more flexible scale invariant Sobolev inequalities in conjunction with parabolic Harnack inequalities. As we have said before there are theorems in different contexts that go along the lines sketched above, we mention [Chr91, HS] for nilpotent Lie groups, [Ale01] and [Ale94] for Hörmander-Mikhlin theorems in discrete groups and Lie groups of polynomial growth and [Heb95] for the general case of metric-measure spaces.

If we want to generalize such theorem to the noncommutative setting we need a natural substitute of Markovian semigroups. We will take all u.c.p and trace preserving semigroups of Fourier multipliers $\lambda_g \mapsto e^{-t\,\psi(g)}\,\lambda_g$, see Section 1.6. As we will see such a semigroup of multipliers is Markovian iff $\psi : G \to \mathbb{R}_+$ is conditionally negative. It is clear that $m(A) = T_{m\circ\psi}$. Formulating Sobolev inequalities in this noncommutative context is relatively easy. The trouble is that neither Step 1 nor Step 3 above work in the noncommutative setting. In order to fix Step 1 we start imposing Gaussian bounds to our semigroup. Even if, very recently, the first steps towards the formulation of a noncommutative Calderón-Zygmund theory have been taken in [Par09, GPJP16, JMP14b], such theory doesn't have yet an scope large enough to cover our situation. To overcome such difficulty we will use a principle of boundedness of Fourier multipliers by maximal functions. The maximal function appearing will be comparable to the Hardy-Littlewood maximal function. In the classical case the boundedness of such maximal operator follows from the doubling condition. An analogue of that theorem does not seems to be true in the noncommutative setting. Therefore, we have to impose it in our hypothesis. When the Gaussian bounds, the doubling condition and the Hardy-Littlewood maximal inequality hold we will say that our space satisfies the standard assumptions. For such spaces we manage to prove a Hörmander-Mikhlin theorem with smoothness controlled by $D/2$, where D is the dimension given by Sobolev inqualities. As an example we recover a result along the lines of the main theorem in [JMP14a].

After a brief review of the different tools that we are going to employ in this document in Chapter 1, we expose the principle of boundedness by maximal operators in Chapter 2. As a quick application, besides the Hörmander-Miklin theorem, is a Marcinkiewicz type multiplier in polynomial Lie Groups that we exposed Section 2.2. The Hörmander-Mikhlin theorems are proved in Chapter 3.

Transference for crossed products

The main results of this block will be contained in Chapter 4. Let G be a group and $\theta : G \to \mathrm{Aut}(\mathcal{M})$ be a normal action preserving the trace $\tau : \mathcal{M}_+ \to [0,\infty]$. Then, it is well known that we can lift the trace τ to a weight on the crossed product algebra $\mathcal{M} \rtimes_\theta G$ and that such weight is tracial if the group is unimodular. Once we have such weight at our disposal it is natural to study the L_p spaces over $\mathcal{M} \rtimes_\theta G$ and whether Fubini type properties hold in this case. Our goal here will be to generalize earlier results of Neuwirth and Ricard [NR11] and Caspers and de la Salle [CdlS15] yielding that for amenable groups the complete boundedness of the Herz-Schur multiplier $M_m : S_p(L_2(G)) \to S_p(L_2(G))$, given by

$$M_m([a_{g\,h}]) = [m(g\,h^{-1})\,a_{g\,h}],$$

where $S_p(L_2 G) = L_p(\mathcal{B}(L_2 G), \mathrm{Tr})$ are the so-called Schatten classes, imply the complete boundedness of the Fourier multiplier $T_m : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)$. The proof uses an asymptotic embedding $j_{p,\alpha} : L_p(\mathcal{L}G) \to S_p(L_2(G))$ obtained from a Følner sequence and ultraproduct techniques to pass to the limit. Here, we generalize such results from amenable groups to amenable actions and crossed products. Indeed, if θ is amenable, then we can construct a completely isometric embedding

$$L_p(\mathcal{M} \rtimes_\theta G) \xrightarrow{j_p} \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))$$

intertwining $\mathrm{Id} \rtimes T_m$ with $\mathrm{Id} \otimes M_m$. Therefore, since it is well-known that $\|M_m\|_{\mathrm{cb}} \leq \|T_m\|_{\mathrm{cb}}$ we get that

$$\|\mathrm{Id} \rtimes T_m\|_{\mathrm{cb}} \leq \|\mathrm{Id} \otimes M_m\|_{\mathrm{cb}} \leq \|T_m\|_{\mathrm{cb}}.$$

One advantage that this method has is that, besides Fourier multipliers, it also allows us to bound crossed product extensions of operators of the form $S \rtimes \mathrm{Id}$, where $S : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is some completely bounded and θ-equivariant operator. Indeed, provided that the approximating sequences associated to the amenable action satisfy a minor accretivity condition we get that

$$\|S \rtimes \mathrm{Id}\|_{\mathrm{cb}} \leq C^{\frac{1}{p}} \|S\|_{\mathrm{cb}},$$

where the constant $1 \leq C$ measures the accretivity. The first application of such result is the crossed-product stability of maximal bounds over noncommutative L_p-spaces, which in turn implies the stability of the standard hypothesis of Chapter 3 under crossed products. Observe that, since our theorem does not require the group G to be amenable it opens the door for potential application to more general exact groups that are discussed in Section 4.4.

W^*-metrics and ideals in $\mathcal{M} \otimes_{eh} \mathcal{M}$

We have already spoken about the fact that in the classical theory the Gaussian bounds can be obtained from Sobolev type inequalities, —in particular ultracontractivity— and either finite speed of propagation or locality for the generator of the semigroup. The results that we summarize here can be understood as a first step towards proving such correspondence in the noncommutative setting. We study a notion of noncommutative metric in the von Neumann algebra language called W^*-metric, defined recently by Kuperberg and Weaver [Wea12, KW12], that seems particularly well-suited for that end. The notion of W^* metric is built on top of that of quantum relations. A quantum relation over $\mathcal{M} \subset \mathcal{B}(H)$ is a weak-$*$ closed $\mathcal{M}'$-bimodule $\mathcal{V} \subset \mathcal{B}(H)$. The intuition here comes from the abelian case $\mathcal{M} = L_\infty(X)$. In such case, the operators $T \in \mathcal{V}$ in a $L_\infty(X)$-bimodule can be represented as integral kernels supported is a measurable set $R \subset X \times X$. A W^*-metric is then a bundle of quantum relations $\mathbb{V} = (\mathcal{V}_t)_{t \geq 0}$ satisfying properties analogous to the triangular inequality, the symmetry and other defining properties for a metric.

Among other characterizations Weaver proved that, in the case of finite dimensional $\mathcal{M}$, quantum relations are in one-to-one correspondence with projections on $\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}$. A similar result can be proved in the case of infinite dimensional algebras $\mathcal{M}$ just by changing the spatial tensor product by the extended Haagerup tensor $\mathcal{M} \otimes_{eh} \mathcal{M}$ and projection by (weakly closed) left ideals. We refer to [Smi91, BS92a] for details on the construction of the Haagerup tensor product. That tensor product is isomorphic to the algebra of c.b. $\mathcal{M}'$-bimodular operators through the map $\Phi : \mathcal{M} \otimes_{eh} \mathcal{M} \to \mathcal{CB}^\sigma_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$ given by extension of

$$\Phi_{x \otimes y}(T) = x\, T\, y.$$

We obtain a bijection between quantum relations and weak-$*$ closed left ideals of $\mathcal{M} \otimes_{eh} \mathcal{M}$. The bijection is nothing but a double annihilator theorem between $\mathcal{M}'$-bimodules and $\mathcal{M}'$-bimodular operators. In the case of a quantum relation $\mathcal{R}$ over $\mathcal{L}G$ that is invariant under the natural comultiplication of $\mathcal{L}G$ we can reduce the ideal to a left ideal inside MG, the algebra of finite Borel measures. Observe that the passing from a tensor algebra $\mathcal{L}G \otimes_{eh} \mathcal{L}G$ to a algebra with a single tensor component is just a way of representing the elimination of one variable, when we work with invariant objects in $G \times G$.

At the end of the chapter, in Section 5.5 we explain the connection with Gaussian bounds and how such ideals can be used to describe off-diagonal restrictions of operators.

Chapter 1

Prerequisites

We will briefly review, pointing out to the references in the literature when necessary, a few tools and constructions that will be used recurrently throughout the text.

1.1 Operator spaces

Through this PhD thesis we are going to employ recurrently the language of operator spaces. We consider it convenient to briefly review some of the basic results and definitions is this subsection, pointing to the references when necessary. More information about operator spaces is now available to the reader in several books like [Pis03, ER00] or the first chapter of [BM04]. An *operator space* is just a closed linear subset $E \subset \mathcal{B}(H)$. Given two operator spaces $E \subset \mathcal{B}(H_1)$ and $F \subset \mathcal{B}(H_2)$ we say that a linear map $\phi : E \to F$ is *completely bounded*, or *c.b.* in short, iff the matrix amplifications $\mathrm{Id} \otimes \phi : M_n[E] \subset \mathcal{B}(\ell_2^n \otimes_2 H_1) \to M_n[F] \subset \mathcal{B}(\ell_2^n \otimes_2 H_2)$ are uniformly bounded on n. We are going to denote by $\mathcal{CB}(E, F)$ the space of all completely bounded (or c.b.) operators. Such space is a Banach space with norm given by

$$\|\phi\|_{\mathrm{cb}} = \sup_{n \geq 1} \big\{ \|\mathrm{Id} \otimes \phi : M_n[E] \to M_n[F]\| \big\}.$$

This quantity will be alternatively denoted by $\|u\|_{\mathcal{CB}(E,F)}$ or $\|u : E \to F\|_{\mathrm{cb}}$. The category of operator spaces is the collection of all operator spaces with c.b. maps as morphisms. There are similar categories that can be defined. One of such is the category of *operators systems*, consisting of all unital and self-adjoint operator spaces, i.e. operator spaces $E \subset \mathcal{B}(H)$ such that $E^* = E$ and $\mathbf{1} \in E$. In such spaces we can define the self adjoint and positive parts $E_+ \subset E_{\mathrm{s.a.}} \subset E$ naturally as the intersection of E with the positive (resp. self-adjoint) part of $\mathcal{B}(H)$ and it is trivial to see that they are nontrivial. The morphisms of this category are the, so-called, *completely positive* maps. A linear map $\phi : E \to F$ is called completely positive (or *c.p.* in short) when $\mathrm{Id}_{M_m} \otimes u$ is positivity preserving for $m \geq 1$. When a c.p. map $\phi : E \to F$ is contractive (resp. unital) we will say it is a c.c.p. (resp. u.c.p.) map. The Kadison-Schwartz inequality for a c.c.p. map $u : \mathcal{M} \to \mathcal{M}$ claims that

$$u(x)^* u(x) \leq u(x^* x) \quad \text{for all} \quad x \in \mathcal{M}.$$

1.1.1 Intrinsic characterization

It was proved in Z-J. Ruan's thesis [ER88] that there is an intrinsic characterization of operator spaces as Banach spaces endowed with collections of matrix norms satisfying the, so called, Ruan's Axioms. Indeed, let E be a Banach space and $\{\|\cdot\|_{M_n[E]}\}_n$, or simply $\{\|\cdot\|_n\}_n$, be a collection of norms over $M_n(E) = M_n \otimes_{\mathrm{alg}} E$. If they satisfy that

(R1) $\|\alpha\, x\, \beta\|_n \leq \|\alpha\|\, \|x\|_n\, \|\beta\|$, for every $x \in M_n[E]$ and $\alpha, \beta \in M_n(\mathbb{C})$,

(R2) For every $x \in M_n[E]$ and $y \in M_m[E]$ we have that

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{n+m} \leq \max\left\{ \|x\|_n, \|y\|_m \right\},$$

then we will say that they satisfy the *Ruan's Axioms*. It is trivial to verify that for any operator space $E \subset \mathcal{B}(H)$ the matrix norms inherited by $M_n[E] \subset \mathcal{B}(\ell_2^n \otimes_2 H)$ satisfy **(R1)** and **(R2)** Axioms. Reciprocally for any Banach spaces endowed with a collection of matrix norms $\{\|\cdot\|_n\}_{n\geq 0}$ there is an isometric embedding $\rho : E \to \mathcal{B}(H)$ whose induced matrix norms coincide with those on the collection, see [BM04, Theorem 1.2.13]. Either an isometric injection $\rho : E \to \mathcal{B}(H)$ or a family of compatible matrix norm will be called an *operator space structure*, or *o.s.s.* in short. If needed, we will denote by E^ρ the operator space whose o.s.s. is given by ρ. The interest of this category is that it lies in between Banach spaces and operator algebras. As we will see below it is, like Banach spaces, flexible enough to be closed under natural constructions —like tensor products, duals, complex interpolation, etc— but, like C^*-algebras, rigid enough to have factorization theorems. Indeed, such factorization theorems precede the birth of operator spaces and appeared as early as [Sti55]. We are going to use in the Chapter 5 the following factorization result due to Wittstock [Wit81] that builds in earlier works of Haagerup and Paulsen.

Theorem 1.1.1 ([BM04, Theorem 1.2.8]). *Let $E \subset \mathcal{A} \subset \mathcal{B}(H)$ be an operator space and $\mathcal{A}$ some C^*-algebra containing it. If $\phi : E \to \mathcal{B}(H_1, H_2)$ is c.b, then, there exists a Hilbert space L, a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(L)$ and bounded operators $S : L \to H_1$ and $T : H_2 \to L$, such that*

$$\phi(x) = S\, \pi(x)\, T$$

for all $x \in X$. Moreover this can be done with $\|S\|\, \|T\| = \|\phi\|_{\mathrm{cb}}$.

1.1.2 A few constructions

Trivially any C^* or von Neumann algebra has a natural o.s.s. give by any faithful, resp. normal and faithful, $*$-homomorphism into $\mathcal{B}(H)$. But, there are much more examples, for instance if E is any Banach space we can faithfully embed it inside $C(\mathrm{Ball}(E^*))$, the space of continuous functions over the unit ball of E^* endowed with the weak-$*$ topology. The induced o.s.s. is called the minimal one and it is easy to see that $\|\phi : F \to E^{\min}\|_{\mathrm{cb}} = \|\phi : F \to E\|$. Similarly the isometric embedding

$$E \hookrightarrow \bigoplus_{m\geq 1} C(\mathrm{Ball}(\mathcal{B}(E, M_m)); M_m),$$

induces the maximal operator space structure, $E^{\max}$. Such o.s.s. satisfies the dual property $\|\phi : E^{\max} \to F\|_{\mathrm{cb}} = \|\phi : E^{\max} \to F\|$. We are also going to work with several o.s.s. over Hilbert spaces, two of them are the so called row and column o.s.s. given by the embeddings

$$\begin{aligned} r(e_j) &= e_{1\,j} \\ c(e_j) &= e_{j\,1}, \end{aligned}$$

where $\{e_j\}_j$ is a basis of ℓ_2 and $\{e_{i\,j}\}_{i,j}$ is a set of matrix units for $\mathcal{B}(H)$.

Recall also that if $\mathcal{CB}(E,F)$ is the space of all completely bounded maps, we may give a natural o.s.s. to such space by isometrically identifying $M_n[CB(E,F)]$ with $\mathcal{CB}(E, M_n[F])$, verifying that such matrix norms satisfy **(R1)** and **(R2)** and applying Ruan's theorem. Since the o.s.s. of $\mathbb{C}$ is trivially minimal, we have that $E^* = \mathcal{B}(E,\mathbb{C}) = \mathcal{CB}(E,\mathbb{C})$ and so the dual of an operator space carries a natural o.s.s. The same can be said about quotients. Analogously we can define complex interpolation of operator spaces. Indeed, if we assume that (X_0, X_1) is a compatible pair of Banach spaces then, $(M_n[X_0], M_n[X_1])$ is also a compatible pair and

$$M_n\big[[X_0, X_1]_\theta\big] = \big[M_n[X_0], M_n[X_1]\big]_\theta,$$

see [BL12] for more on interpolation and [Pis03, pp. 112] for the precise definitions in the context of operator spaces.

Operator spaces also admit a theory of tensor products analogous to that of Banach spaces, see [Rya13]. In particular there are infinitely many well-behaved tensor products and among them the smallest is the *projective tensor product*, that we will denote by $\widehat{\otimes}$, and the largest one is the injective or *minimal tensor product*, denoted by $\otimes_{\mathrm{min}}$. It is worth recalling that such products do not coincide in general with the Banach space projective or injective tensor products. In contrast with what happens in the category of Banach spaces, the category of operator spaces has a tensor product which is simultaneously injective and projective, the *Haagerup tensor product* that will be treated more extensively in Chapter 5.

1.2 Noncommutative L_p-spaces

Part of von Neumann algebra theory has evolved as the noncommutative form of measure theory and integration. A von Neumann algebra $\mathcal{M}$ [KR97, Tak79, Tak03], is a unital weak-operator closed C^*-subalgebra of $\mathcal{B}(H)$, the algebra of bounded linear operators on a Hilbert space H. We will write $\mathbf{1}_{\mathcal{M}}$, or simply $\mathbf{1}$, for the unit. The positive cone $\mathcal{M}_+$ is the set of positive operators in $\mathcal{M}$ and a trace $\tau : \mathcal{M}_+ \to [0,\infty]$ is a linear map satisfying $\tau(x^*x) = \tau(xx^*)$. Such map is said to be:

1. *normal* iff for every increasing net of projections $(p_\alpha)_\alpha \subset M_+$, $\sup_\alpha \tau(p_\alpha) = \tau(\sup_\alpha p_\alpha)$,
2. *semifinite* iff for $x \in \mathcal{M}_+ \setminus \{0\}$ there exists $0 < x' \leq x$ with $\tau(x') < \infty$,
3. *σ-finite* iff there is a numerable family of projections $\{p_j\}_j$ s.t. $\tau(p_j) < \infty$ and $\mathbf{1} = \sum_{j=0}^{\infty} p_j$,
4. *faithful* iff for every $x \in M_+$, $\tau(x) = 0$ implies that $x = 0$.

When a trace is normal, semifinite and faithful we will say it is a *n.s.f.* trace. The trace τ plays the role of the integral in the classical case. A von Neumann algebra $\mathcal{M}$ is *semifinite* when it admits a normal semifinite faithful trace τ. Any $x \in \mathcal{M}$ is a linear combination $x_1 - x_2 + ix_3 - ix_4$

of four positive operators. Thus, τ extends as an unbounded operator to nonpositive elements and the tracial property takes the familiar form $\tau(xy) = \tau(yx)$. The pairs $(\mathcal{M}, \tau)$ composed by a von Neumann algebra and a n.s.f. trace will be called *noncommutative measure spaces*. Note that commutative von Neumann algebras correspond to classical measurable spaces.

By the GNS construction, the noncommutative analogue of measurable sets (characteristic functions) are orthogonal projections. Given $x \in \mathcal{M}_+$, its support is the least projection q in $\mathcal{M}$ such that $qx = x = xq$ and is denoted by supp$[x]$. Let $S_{\mathcal{M}}^+$ be the set of all $x \in \mathcal{M}_+$ such that $\tau(\mathrm{supp}[x]) < \infty$ and set $S_{\mathcal{M}}$ to be the linear span of $\mathcal{S}_{\mathcal{M}}^+$. If we write $|x| = \sqrt{x^*x}$, we can use the spectral measure dE of $|x|$ to observe that for every $x \in S_{\mathcal{M}}$

$$|x|^p = \int_{\mathbb{R}_+} s^p \, dE(s) \in S_{\mathcal{M}}^+ \Rightarrow \tau(|x|^p) < \infty.$$

If we set $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$, we obtain a norm in $S_{\mathcal{M}}$ for $1 \leq p < \infty$. By the strong density of $S_{\mathcal{M}}$ in $\mathcal{M}$, the *noncommutative* L_p *space* $L_p(\mathcal{M})$ is the corresponding completion for $p < \infty$ and $L_\infty(\mathcal{M}) = \mathcal{M}$. Many basic properties of classical L_p spaces like duality, real and complex interpolation, Hölder inequalities, etc hold in this setting. Elements of $L_p(\mathcal{M})$ can be described as measurable operators affiliated to $(\mathcal{M}, \tau)$, we refer to Pisier/Xu's survey [PX03] for more information and historical references. Note that classical L_p spaces $L_p(\Omega, \mu)$ are denoted in this terminology as $L_p(\mathcal{M})$ where $\mathcal{M}$ is the commutative von Neumann algebra $L_\infty(\Omega, \mu)$.

Now, we proceed to describe the natural o.s.s. of $L_p(\mathcal{M})$. Given an operator space E, its opposite E_{op} is the operator space which comes equipped with the operator space structure determined by the o.s.s. of E as follows

$$\Big\| \sum_{j,k=1}^m a_{jk} \otimes e_{jk} \Big\|_{M_m(E_{\mathrm{op}})} = \Big\| \sum_{j,k=1}^m a_{kj} \otimes e_{jk} \Big\|_{M_m(E)},$$

where e_{jk} stand for the matrix units in M_m. Alternatively, if $E \subset \mathcal{B}(H)$, then $E_{\mathrm{op}} = E^\top \subset \mathcal{B}(H)$, where $\top$ is the transpose. The op construction plays a role in the construction of a "natural" o.s.s. for noncommutative L_p spaces. If $\mathcal{M}$ is a von Neumann algebra we will denote by $\mathcal{M}_{\mathrm{op}}$ its opposite algebra, the original algebra with the multiplication reversed. It is a well-known result that $\mathcal{M}_{\mathrm{op}}$ and $\mathcal{M}$ need not be isomorphic [Con75]. Of course the natural o.s.s. of $\mathcal{M}_{\mathrm{op}}$ coincide with the opossite o.o.s. of $\mathcal{M}$. For every operator space E the natural inclusion $j : E \to E^{**}$ is a complete isometry. This allows us to build an operator space structure in the predual $\mathcal{M}_*$ as the only operator space structure that makes the inclusion $j : \mathcal{M}_* \to \mathcal{M}^*$ completely isometric. The operator space structure of $L_p(\mathcal{M})$ is given by operator space complex interpolation between the following two endpoints

$$\begin{array}{rcl} L_1(\mathcal{M}) & = & \mathcal{M}_*^{\mathrm{op}} \\ L_\infty(\mathcal{M}) & = & \mathcal{M}. \end{array}$$

In particular, it turns out that $L_p(\mathcal{M})^* \simeq L_{p'}(\mathcal{M}_{\mathrm{op}})$ is a complete isometry for $1 \leq p < \infty$, see [Pis03, pp. 120-121] for further details. Such spaces generalize the classical L_p-spaces $L_p(X, \mu)$ when $\mathcal{M} = L_\infty(X)$ and there are isomorphic identifications of $L_\infty(\mathcal{M})$ with $\mathcal{M}$, of $L_2(\mathcal{M})$ with the GNS construction of τ and of $L_1(\mathcal{M})$ with the predual $\mathcal{M}_*$.

1.3 Group von Neumann algebras

Let G be a locally compact and Hausdorff (LCH in short) group. It is well known that LCH groups admit unique left or right invariant measures and that both coincide if the group is unimodular. We denote by μ the Haar measure and by $L_p(G) = L_p(G, \mu)$ the L_p spaces of functions associated with that measure. From this point onward we will assume G to be second-countable, i.e. to have a numerable base for the topology. Let $\lambda : G \to \mathcal{U}(L_2 G)$ be the left regular representation. It is easy to check that such representation is SOT-continuous. Given a SOT-continuous unitary representation π we will denote by $\pi(f)$ the linear extension of π to $f \in L_1(G)$, i.e:

$$\lambda(f) = \int_G f(g)\, \lambda_g \, d\mu(g),$$

and by $\pi(\mu)$ the extension to any finite Radon measure $\nu \in M(G)$, defined similarly. The *(reduced) group C^* algebra* and *group von Neumann* of G are given by

$$\begin{array}{rcl} C^*_\lambda G & = & \overline{\{\lambda(f) \in \mathcal{B}(L_2 G) : f \in L_1(G)\}}^{\|\cdot\|_{\mathcal{B}(L_2 G)}} \\ \mathcal{L}G & = & (C^*_\lambda G)''. \end{array}$$

The intuition on $\mathcal{L}G$ or $C^*_\lambda G$ is that, whenever G is an abelian LCH group, $C^*_\lambda G$ is an abelian C^*-algebra $*$-isomorphic, by Gel'fand's theorem, to the C^*-algebra of continuous functions vanishing at infinity over the dual group $\hat{G}$, see [Fol95] or [Rud90]. Therefore, $\mathcal{L}G$ can be regarded as a noncommutative generalization of the algebra of essentially bounded functions over the dual.

There is a distinguished normal faithful weight $\tau : \mathcal{L}G_+ \to \mathbb{R}_+$ such that $\lambda : L_1(G) \cap L_2(G) \to \mathcal{L}G$ extends to an isometry from $L_2(G)$ to $L_2(\mathcal{L}G, \tau)$, the GNS construction associated to τ. Such weight is unique and it is called the *Plancherel weight*, see [Ped79, Chapter 7]. When the function f belongs to the dense class $C_c(G) * C_c(G)$ we have

$$\tau(\lambda(f)) = f(e).$$

The Placherel weight is tracial if and only if G is unimodular. In this case it is called the Placherel trace. From now on we will focus on unimodular groups. We will often work with the spaces $L_p(\mathcal{L}G, \tau)$ although the dependency on τ will be dropped in our terminology.

$\mathcal{L}G$ has a natural comultiplication given by linear extension of $\delta(\lambda_g) = \lambda_g \otimes \lambda_g$ which extends to a $*$-homomorphism $\delta : C^*_\lambda G \to C^*_\lambda G \otimes_{\min} C^*_\lambda G$. There is a unique normal extension $\delta : \mathcal{L}G \to \mathcal{L}G \,\overline{\otimes}\, \mathcal{L}G$. This is a consequence of the fact that if δ is normal then $\delta_* : \mathcal{L}G_* \,\widehat{\otimes}\, \mathcal{L}G_* \to \mathcal{L}G_*$. Here $\otimes_{\min}$ and $\widehat{\otimes}$ represent respectively the minimal and projective o.s. tensor products [Pis03] and $\overline{\otimes}$ denotes the weak operator closure of the algebraic tensor product. Identifying $\mathcal{L}(G \times G)_*$ with $\mathcal{L}G_* \,\widehat{\otimes}\, \mathcal{L}G_*$ we have

$$\delta_* \Big(\int_{G \times G} f(g_1, g_2)\, \lambda_{(g_1, g_2)}\, d\mu(g_1) d\mu(g_2) \Big) = \int_G f(g, g)\, \lambda_g \, d\mu(g),$$

for every $f \in C_c(G \times G) * C_c(G \times G)$. The boundedness of δ_* is then a consequence of the Herz restriction theorem [Her72]. It is interesting to note that the Plancherel weight can be characterized as the unique normal, nontrivial and δ-invariant weight, where δ-invariant means that

$$(\tau \otimes \mathrm{Id})\, \delta x = \tau(x)\, \mathbf{1}.$$

Analogously, Fourier multipliers are characterized as δ-equivariant maps

$$\delta\, T = (T \otimes \mathrm{Id})\, \delta = (\mathrm{Id} \otimes T)\, \delta.$$

We will denote by $\sigma : \mathcal{L}G \to \mathcal{L}G$ the anti-automorphism given by linear extension of $\sigma(\lambda_g) = \lambda_{g^{-1}}$. The *quantized convolution* of two elements x, y affiliated to $\mathcal{L}G$ is defined by

$$x \star y = (\tau \otimes \mathrm{Id})\big\{\delta x \, (\sigma y \otimes \mathbf{1})\big\}.$$

Observe that given $m \in L_\infty(G)$, the corresponding Fourier multiplier has the form

$$T_m(x) = \lambda(m) \star x = (\tau \otimes \mathrm{Id})\big\{\delta \lambda(m) \, (\sigma x \otimes \mathbf{1})\big\}.$$

1.4 Vector-valued L_p-spaces

We will denote by S_p the Schatten p-class given by $S_p = L_p(\mathcal{B}(\ell_2), \mathrm{Tr})$ with Tr the standard trace in $\mathcal{B}(\ell_2)$. Similarly, S_p^m stands for the same space over $m \times m$ matrices. Vector-valued forms of these spaces can be defined as long as we define an o.s.s. over the space where we take values. Given an operator space E, we may define the *E-valued Schatten classes* $S_p^m[E]$ as the operator spaces given by interpolation

$$S_p^m[E] \, := \, \big[S_\infty^m[E], S_1^m[E]\big]_{\frac{1}{p}} \, := \, \big[S_\infty^m \otimes_{\min} E, S_1^m \, \widehat{\otimes} \, E\big]_{\frac{1}{p}}.$$

These classes provide a useful characterization of complete boundedness.

Lemma 1.4.1 ([Pis98, Lemma 1.7]). *For every completely bounded $\phi : E \to F$ we have that*

$$\|\phi\|_{\mathcal{CB}(E,F)} = \sup_{m \geq 1} \Big\{ \big\|\mathrm{Id}_{M_m} \otimes \phi : S_p^m[E] \to S_p^m[F]\big\| \Big\},$$

for $1 \leq p \leq \infty$.

For a general hyperfinite von Neumann algebra $\mathcal{M}$ the construction of $L_p(\mathcal{M}; E)$ is carried out by direct limits of E-valued Schatten classes. We refer to Pisier's book [Pis98] for more on vector-valued noncommutative L_p spaces. The space $L_p(\mathcal{M}; E)$ for nonhyperfinite $\mathcal{M}$ cannot be constructed without losing fundamental properties like projectivity/injectivity of the functor $E \mapsto L_p(\mathcal{M}; E)$. Fortunately, this drawback is solvable for the vector-valued L_p space we shall be working with.

1.5 Hilbert-valued L_p-spaces

For certain operator spaces whose underlying Banach space is a Hilbert space we can define vector-valued noncommutative L_p spaces for general von Neumann algebras. Indeed, let H be a Hilbert space and and $P_e \xi = \langle e, \xi \rangle e$ for some $e \in H$ of unit norm. We define the following two Hilbert-valued forms of $L_p(\mathcal{M})$

$$\begin{array}{rcl} L_p(\mathcal{M}; H^c) & = & L_p(\mathcal{M} \, \overline{\otimes} \, \mathcal{B}(H))(\mathbf{1}_\mathcal{M} \otimes P_e), \\ L_p(\mathcal{M}; H^r) & = & (\mathbf{1}_\mathcal{M} \otimes P_e) L_p(\mathcal{M} \, \overline{\otimes} \, \mathcal{B}(H)), \end{array}$$

called the L_p spaces with H-column (resp. H-row) values. Their o.s.s. are the ones inherited from $L_p(\mathcal{M} \, \overline{\otimes} \, \mathcal{B}(H))$. If $H = \ell_2^n$, then we can identify $L_p(\mathcal{B}(H) \, \overline{\otimes} \, \mathcal{M})$ with $L_p(\mathcal{M})$-valued $n \times n$

matrices. Under that identification $L_p(\mathcal{M}; H^c)$ (resp. $L_p(\mathcal{M}; H^r)$) corresponds to the matrices with zero entries outside the first column (resp. row) and we have that

$$\begin{aligned} \Big\| \sum_{j=1}^n x_j \otimes e_{j1} \Big\|_{L_p(\mathcal{M} \otimes \mathcal{B}(\ell_2^n))} &= \Big\| \Big(\sum_{j=1}^n x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \\ \Big\| \sum_{j=1}^n x_j \otimes e_{1j} \Big\|_{L_p(\mathcal{M} \otimes \mathcal{B}(\ell_2^n))} &= \Big\| \Big(\sum_{j=1}^n x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}. \end{aligned}$$

The same formulas hold after replacing the finite sums by infinite ones of even by integrals. For every $1 \leq p \leq \infty$ we can embed H isometrically in S_p by sending $c_p(e_j) = e_{1\,j}$ or $r_p(e_j) = e_{j\,1}$, where $\{e_j\}$ is an orthonormal basis of H. Such maps are called the p-column/p-row embedings. These isometries endow H with several o.s. structures. Observe that, as an o.s, $L_p(\mathcal{M}; H^c)$ (resp. $L_p(\mathcal{M}; H^r)$) coincides with Pisier's vector-valued L_p-space $L_p(\mathcal{M}; H^{c_p})$ (resp. $L_p(\mathcal{M}; H^{r_p})$) for $\mathcal{M}$ hyperfinite. For $1 \leq p < \infty$ the duals are given by $L_p(\mathcal{M}; H^c)^* = L_{p'}(\mathcal{M}_{\mathrm{op}}; H^c)$ and $L_p(\mathcal{M}; H^c)^* = L_{p'}(\mathcal{M}_{\mathrm{op}}, H^c)$. The duality pairing can be expres as

$$\Big\langle \sum\nolimits_j x_j \otimes e_j, \sum\nolimits_k y_k \otimes e_k \Big\rangle = \sum\nolimits_j \tau(x_j^* y_j).$$

The spaces $L_p(\mathcal{M}; H^r)$ and $L_p(\mathcal{M}; H^c)$ form complex interpolation scales for $p \geq 1$

$$\begin{aligned} \left[L_\infty(\mathcal{M}; H^r), L_p(\mathcal{M}; H^r) \right]_\theta &= L_{\frac{p}{\theta}}(\mathcal{M}; H^r), \\ \left[L_\infty(\mathcal{M}; H^c), L_p(\mathcal{M}; H^c) \right]_\theta &= L_{\frac{p}{\theta}}(\mathcal{M}; H^c). \end{aligned}$$

In order to treat square functions and Hardy spaces we will need to control sums and intersections of these Hilbert valued noncommutative L_p spaces. The algebraic tensor product $L_p(\mathcal{M}) \otimes H$ embeds in $L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(H))$ by $\mathrm{Id} \otimes r$ and $\mathrm{Id} \otimes c$. Taking direct sums we obtain an embedding in $X = L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(H)) \oplus L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(H))$. The space $L_p(\mathcal{M}; H^{r \cap c})$ is defined as the norm closure (or weak-$*$ closure if $p = \infty$) of $L_p(\mathcal{M}) \otimes H$ inside X. Such space comes equipped with the norm given by

$$\|x\|_{L_p(\mathcal{M}; H^{r \cap c})} = \max \Big\{ \|x\|_{L_p(\mathcal{M}; H^r)}, \|x\|_{L_p(\mathcal{M}; H^c)} \Big\}.$$

The embedding also gives $L_p(\mathcal{M}; H^{r \cap c})$ an o.s.s. We will denote the dual spaces by $L_p(\mathcal{M}; H^{r+c}) = L_{p'}(\mathcal{M}_{\mathrm{op}}; H^{r \cap c})^*$ for $1 < p \leq \infty$. The space $L_1(\mathcal{M}; H^{r+c})$ is defined as the subset of weak-$*$ continuous functionals in $L_\infty(\mathcal{M}_{\mathrm{op}}; H^{r \cap c})^*$. The sum notation comes from the fact that

$$\|x\|_{L_p(\mathcal{M}; H^{r+c})} = \inf \Big\{ \|y\|_{L_p(\mathcal{M}; H^r)} + \|z\|_{L_p(\mathcal{M}; H^c)} : x = y + z \Big\}.$$

We will denote by $L_p(\mathcal{M}; H^{rc})$ the spaces given by

$$L_p(\mathcal{M}; H^{rc}) = \begin{cases} L_p(\mathcal{M}; H^{r+c}) & \text{when } 1 \leq p < 2 \\ L_p(\mathcal{M}; H^{r \cap c}) & \text{when } 2 \leq p \leq \infty. \end{cases}$$

The spaces $L_p(\mathcal{M}; H^{rc})$ are crucial for the noncommutative Khintchine inequalities [LP86, LPP91], the noncommutative Burkholder-Gundy inequalities [JX03], noncommutative Littlewood-Paley estimates [JLMX06] and other related results.

1.6 Markovian semigroups

Recall some definitions. A semigroup $\mathcal{S} = (S_t)_{t \geq 0}$ over a Banach space X is a family of operators $S_t : X \to X$ such that $S_0 = \mathrm{Id}$ and $S_t S_s = S_{t+s}$. Let $(\mathcal{M}, \tau)$ be a noncommutative measure space, we will say that a semigroup $\mathcal{S}$ over $\mathcal{M}$ is *Markovian* iff:

Definition 1.6.1. A semigroup $(S_t)_{t\geq 0}$ of normal operators $S_t : \mathcal{M} \to \mathcal{M}$ is said to be Markovian iff

(i) Each S_t is unital and completely positive.

(ii) The semigroup is symmetric, i.e. $\tau((S_t x)^* y) = \tau(x^* S_t y)$.

(iii) The map $t \mapsto S_t$ is pointwise weak-$*$ continuous.

Observe that as a consequence of S_t being unital and **(ii)** we get that $\tau \circ S_t = \tau$.

$\mathcal{S}$ is *submarkovian* if each S_t is a c.c.p. map satisfying $\tau \circ S_t \leq \tau$. Sometimes these semigroups are called symmetric and Markovian, where symmetric is synonym with self-adjoint. All the semigroups in this paper will be symmetric, so we will drop the adjective. Using the first two properties it is easy to see that S_t extends to a c.c.p. map on $L_1(\mathcal{M})$. By the Riesz-Thorin theorem S_t is a complete contraction over $L_p(\mathcal{M})$ for $1 \leq p \leq \infty$. The third property implies that $t \mapsto S_t$ is SOT continuous in $L_1(\mathcal{M})$. By interpolation we obtain that $t \mapsto S_t$ is SOT-continuous on $L_p(\mathcal{M})$ for $1 \leq p < \infty$. For every $1 \leq p < \infty$ there is a densely defined and closable operator A whose closed domain is given by

$$\mathrm{dom}_p(A) = \Big\{ x \in L_p(\mathcal{M}) : \exists \lim_{t \to 0^+} \frac{x - S_t x}{t} \text{ in the norm topology } \Big\}.$$

When $p = 2$ we have that $S_t = e^{-tA}$ and $S_t[L_p(\mathcal{M})] \subset \mathrm{dom}_p(A)$ for $1 \leq p < \infty$. In the case $p = \infty$ we have that A is densely defined and closable with respect to the weak-$*$ topology with domain given by those $x \in \mathcal{M}$ such that $\lim_{t\to 0^+}(x - S_t x)/t$ exists in the weak-$*$ topology. We will call A the infinitesimal generator of $\mathcal{S}$.

We are interested in those (sub)markovian semigroups over $\mathcal{M} = \mathcal{L}G$ which are of convolution type. In other words, each S_t is a Fourier multiplier. It can be proved that S_t is of the for $S_t = T_{e^{-t\psi}}$ for some function ψ. Let us recall a characterization of these functions. First, some definitions. A continuous function $\psi : G \to \mathbb{C}$ is said to be *conditionally negative* (or *c.n.* in short) iff $\psi(e) = 0$ and for every finite subset $\{g_1, ..., g_m\} \subset G$ and vector $(v_1, .., v_m) \in \mathbb{C}^m$ we have

$$\sum_{i=1}^m v_i = 0 \quad \Rightarrow \sum_{i,j=1}^m \bar{v}_i \psi(g_i^{-1} g_j) v_j \leq 0.$$

When $\psi : G \to \mathbb{R}_+$ is symmetric ($\psi(g) = \psi(g^{-1})$) and c.n. we will say that ψ is a *conditionally negative length*. Let H be a real Hilbert space. Given an orthogonal representation $\alpha : G \to O(H)$ we say that a continuous map $b : G \to H$ is a 1-*cocycle* (with respect to α) iff it satisfies the 1-cocycle law

$$b(gh) = \alpha(g) b(h) + b(g).$$

The following characterization is proved in [BdlHV08, Appendix C] or [CCJ+01, Chapter 1].

Theorem 1.6.1. *Let $\mathcal{S} = (S_t)_{t\geq 1}$ be a semigroup of convolution type over the group algebra $\mathcal{L}G$. Then, the following statements are equivalent:*

(i) *$\mathcal{S}$ is a Markovian semigroup.*

(ii) *There is a c.n. length $\psi : G \to \mathbb{R}_+$ such that $S_t = T_{e^{-t\psi}}$.*

(iii) *There is a real Hilbert space H, an orthogonal representation $\alpha : G \to O(H)$ and a 1-cocycle $b : G \to H$, such that $\psi(g) = \|b(g)\|_H^2$ and $S_t = T_{e^{-t\psi}}$*

1.7 Bounds for Fourier multipliers

In general, determining the c.b. norm of a multiplier between general L_p spaces is a problem that nobody expects to be solvable with a closed formula. Nevertheless, some cases do admit a closed solution. For example, completely bounded Fourier multipliers acting on $L_1(\mathcal{L}G)$ are classified, when, G is amenable after the works of Fendler and Bożejko, see [BF91]. Similar arguments holds for normal and c.b. multipliers in $\mathcal{L}G$.

Other case in which the c.b. Fourier multipliers can be classified is that of $T_m : L_2(\mathcal{L}G) \to \mathcal{L}G$. Indeed, such multipliers are exactly those with an L_2-symbol. The next theorem is probably known to experts. Since we could not find it in the literature, we include it here for the sake of completeness.

Theorem 1.7.1. *If T denotes the map $m \mapsto T_m$, then*

(1) $T : L_2^r(G) \to \mathcal{CB}(L_2^c(\mathcal{L}G), \mathcal{L}G)$ *is a complete isometry.*

(2) $T : L_2^c(G) \to \mathcal{CB}(L_2^r(\mathcal{L}G), \mathcal{L}G)$ *is a complete isometry.*

The image of T is the set of multipliers $T_m : L_2^\dagger(\mathcal{L}G) \xrightarrow{\mathrm{cb}} \mathcal{L}G$ for $\dagger \in \{c, r\}$ resp.

Proof. Let V and W be operator spaces and pick $x \otimes y \in V^* \otimes W^*$. According to [Pis03, Theorem 4.1] the map $\mathcal{I}_{x\otimes y}(w) = x\langle y, w\rangle$ extends linearly to an isomorphism $\mathcal{I} : (V \mathbin{\widehat{\otimes}} W)^* \to \mathcal{CB}(W, V^*)$. Using the pairing $\langle \, , \rangle : L_2^r(\mathcal{L}G) \times L_2^c(\mathcal{L}G) \to \mathbb{C}$ given by $\langle y, w\rangle = \tau(y\,\sigma w)$ we obtain as a consequence that

$$\mathcal{I}_{\delta z}(w) = (\mathrm{Id} \otimes \tau)\big(\delta z\, (\mathbf{1} \otimes \sigma w)\big) = z \star w,$$

where δz denotes the comultiplication map acting on z. This yields

$$\big\| T_m : L_2^\dagger(\mathcal{L}G) \to \mathcal{L}G \big\|_{\mathrm{cb}} = \big\| \delta\lambda(m) \big\|_{(\mathcal{L}G_* \mathbin{\widehat{\otimes}} L_2^\dagger(\mathcal{L}G))^*}$$

where $\dagger \in \{r, c\}$ is either the row or the column o.s.s. We now claim that the natural map

$$\iota : L_\infty(\mathcal{L}G; L_2^{\dagger^{\mathrm{op}}}(\mathcal{L}G)) \hookrightarrow \big(\mathcal{L}G_* \mathbin{\widehat{\otimes}} L_2^\dagger(\mathcal{L}G)\big)^*$$

is a complete isometry with $\dagger^{\mathrm{op}} = r$ for $\dagger = c$ and viceversa. This is all what is needed to complete the argument since we have the following commutative diagram of complete isometries

$$\begin{tikzcd}
L_2^\dagger(G) \arrow[rr, "T"] \arrow[d, "\lambda"] & & \mathcal{CB}(L_2^{\dagger^{\mathrm{op}}}(\mathcal{L}G), \mathcal{L}G) \, . \\
L_2^\dagger(\mathcal{L}G) \arrow[dr, "\delta"] & & (\mathcal{L}G_* \mathbin{\widehat{\otimes}} L_2^{\dagger^{\mathrm{op}}}(\mathcal{L}G))^* \arrow[u, "\mathcal{I}"] \\
& L_\infty(\mathcal{L}G; L_2^\dagger(\mathcal{L}G)) \arrow[ur, hook, "\iota"] &
\end{tikzcd}$$

Let us therefore justify our claim. According to [ER00]

$$(\mathcal{L}G_* \mathbin{\widehat{\otimes}} L_2^\dagger(\mathcal{L}G))^* = \mathcal{L}G \otimes_{\mathcal{F}} L_2^{\dagger^{\mathrm{op}}}(\mathcal{L}G)$$

where $\otimes_{\mathcal{F}}$ stands for the Fubini tensor product of dual operator spaces. Bear in mind that if V^* and W^* are dual operator spaces, there are weak-$*$ continuous embeddings $V^* \subset \mathcal{B}(H_1)$ and $W^* \subset \mathcal{B}(H_2)$ and we can define the weak-$*$ spatial tensor product $V^* \overline{\otimes} W^*$ as

$$V^* \overline{\otimes} W^* = \overline{(V^* \otimes W^*)^{\mathrm{w}^*}}.$$

Such construction is representation independent and $V^* \overline{\otimes} W^*$ embeds completely isometricaly in $V^* \otimes_{\mathcal{F}} W^*$. Since the column and row embeddings of $L_2(\mathcal{L}G)$ into $\mathcal{B}(L_2(\mathcal{L}G))$ are weak-$*$ continuous, $L_\infty(\mathcal{L}G; L_2^{\dagger^{\mathrm{op}}}(\mathcal{L}G)) = \mathcal{L}G \overline{\otimes} L_2^{\dagger^{\mathrm{op}}}(\mathcal{L}G)$. This proves that ι is a complete isometry and so is the map $m \mapsto T_m = \mathcal{I}_{\iota\delta\lambda(m)}$. □

Remark 1.7.2. Since $L_2^r(\mathcal{L}G)$ and $L_2^c(\mathcal{L}G)$ are isometric as Banach spaces, the norms for multipliers in $\mathcal{CB}(L_2^r(\mathcal{L}G), \mathcal{L}G)$ and $\mathcal{CB}(L_2^c(\mathcal{L}G), \mathcal{L}G)$ coincide too, even if their matrix amplifications do not. Indeed we obtain that

$$\|T_m\|_{\mathcal{CB}(L_2^r(\mathcal{L}G),\mathcal{L}G)} = \|m\|_{L_2(G)} = \|T_m\|_{\mathcal{CB}(L_2^c(\mathcal{L}G),\mathcal{L}G)}.$$

For non-hyperfinite $\mathcal{L}G$, the space of Fourier multipliers in $\mathcal{CB}(L_2(\mathcal{L}G), \mathcal{L}G)$, may be difficult to describe as an operator space. Nevertheless, as a consequence of the above identities, its underlying Banach space is the Hilbert space $L_2(G)$.

1.8 Noncommutative Maximal inequalities

Maximal inequalities are a cornerstone in harmonic analysis. Unfortunately, the supremun of a family of noncommuting operators is not well-defined, so that we do not have a proper noncommutative analogue of maximal functions. Nevertheless, this difficulty can be overcome if all we want is to bound is the maximal function in noncommutative L_p, as usually happens in harmonic analysis for commutative spaces. In that case we exploit the fact that the p-norm of a maximal function can always be written as a mixed $L_p(L_\infty)$-norm of the corresponding entries. This reduces the problem to construct the vector-valued spaces $L_p(\mathcal{M}; L_\infty(\Omega))$. This construction can be carried out without requiring $\mathcal{M}$ to be hyperfinite, relaying in the commutativity of $L_\infty(\Omega)$. $L_p(\mathcal{M}; L_\infty(\Omega))$ is defined as the subspace of functions $x \in L_\infty(\Omega; L_p(\mathcal{M}))$ which admit a factorization of the form $x_\omega = \alpha\, y_\omega\, \beta$ with $\alpha, \beta \in L_{2p}(\mathcal{M})$ and $y \in L_\infty(\Omega; \mathcal{M})$. The norm in such space is then given by

$$\big\|(x_\omega)_{\omega\in\Omega}\big\|_{L_p(\mathcal{M};L_\infty(\Omega))} \, = \, \inf \Big\{ \|\alpha\|_{2p} \Big(\operatorname*{ess\,sup}_{\omega\in\Omega} \|y_\omega\|_{\mathcal{M}} \Big) \|\beta\|_{2p} \, : \, x = \alpha y \beta \Big\}.$$

When $x_\omega \geq 0$ the norm coincides with

$$\big\|(x_\omega)_{\omega\in\Omega}\big\|_{L_p(\mathcal{M};L_\infty(\Omega))} \, = \, \inf \Big\{ \|y\|_{L_p(\mathcal{M})} \, : \, x_\omega \leq y \text{ for a.e. } \omega \in \Omega \Big\}. \tag{1.8.1}$$

Its operator space structure satisfies

$$S_p^m\big[L_p(\mathcal{M}; L_\infty(\Omega))\big] \, = \, L_p\big(M_m \otimes \mathcal{M}; L_\infty(\Omega)\big).$$

It is standard to use the following notation for the noncommutative $L_p(L_\infty)$-norm

$$\Big\| \sup_{\omega\in\Omega}{}^+ \, x_\omega \Big\|_{L_p(\mathcal{M})} = \big\|(x_\omega)_{\omega\in\Omega}\big\|_{L_p(\mathcal{M};L_\infty(\Omega))},$$

where the sup is just a symbolic notation without an intrinsic meaning. In the proof of Theorem 3.2.1 we will use the fact that if $(\mu_{\omega_2})_{\omega_2\in\Omega_2}$ is a family of finite positive measures in Ω_1 and

$(R_{\omega_1})_{\omega_1\in\Omega_1}$ is a family of positivity preserving operators, then the following bound holds for $x \in L_p(\mathcal{M})_+$

$$\Big\| \sup_{\omega_2\in\Omega_2}{}^{+} \Big\{ \int_{\Omega_1} R_{\omega_1}(x) d\mu_{\omega_2}(\omega_1) \Big\} \Big\|_p \leq \Big(\sup_{\omega_2\in\Omega_2} \|\mu_{\omega_2}\|_{M(\Omega)} \Big) \Big\| \sup_{\omega_1\in\Omega_1}{}^{+} R_{\omega_1}(x) \Big\|_p. \tag{1.8.2}$$

When $\mathcal{M}$ is hyperfinite, this definition of $L_p(\mathcal{M}; L_\infty(\Omega))$ coincides with the corresponding vector-valued space as defined by Pisier [Pis98]. This approach to handle maximal inequalities in von Neumann algebras has been successfully used in [Jun02] to find noncommutative forms of Doob's maximal inequality for martingales and the maximal ergodic inequalities for Markov semigroups [JX07]. The predual can be explicitly described as the L_1-valued space $L_{p'}(\mathcal{M}; L_1(\Omega))$. Indeed, let $S_p(\Omega)$ be the Schatten class associated to the Hilbert space $L_2(\Omega)$. Note that there is a hermitian form $q : L_{2p}(\mathcal{M}) \otimes S_2^c(\Omega) \times L_{2p}(\mathcal{M}) \otimes S_2^c(\Omega) \to L_p(\mathcal{M}) \otimes L_1(\Omega)$ given by

$$q(x \otimes m, y \otimes n) = x^* y \otimes \mathrm{diag}(m^* n),$$

where $\mathrm{diag} : S_1(\Omega) \to L_1(\Omega)$ is the restriction to the diagonal. Define

$$\|x\|_{L_p(\mathcal{M};L_1(\Omega))} = \inf \Big\{ \|a\|_{L_{2p}(\mathcal{M};S_2^c(\Omega))} \|b\|_{L_{2p}(\mathcal{M};S_2^c(\Omega))} : q(a,b) = x \Big\}.$$

This space satisfies that $L_p(\mathcal{M}; L_1(\Omega))^* = L_{p'}(\mathcal{M}_{\mathrm{op}}; L_\infty(\Omega))$ for $1 \leq p < \infty$.

1.9 $\mathcal{H}^\infty$-calculus

We now introduce the Hardy spaces associated with a submarkovian semigroup on $(\mathcal{M}, \tau)$ as well as the corresponding $\mathcal{H}^\infty$-functional calculus. Both tools were introduced in the noncommutative setting in [JLMX06]. If $\mathcal{S}$ is a submarkovian semigroup, the fixed point subspace $F_p = \{x \in L_p(\mathcal{M}) : S_t(x) = x \ \forall\, t \geq 0\}$ coincides with $\ker A \subset \mathrm{dom}_p(A)$ and it is a subalgebra when $p = \infty$. It is also easily seen to be a complemented subspace with projection given by $Q_p(x) = \lim_{t\to\infty} S_t x$ where the limit converges in the norm topology of L_p, for $p < \infty$ and in the weak-$*$ topology when $p = \infty$. We will denote by $L_p^\circ(\mathcal{M}) = L_p(\mathcal{M})/F_p$ which is also a complemented subspace with projection given by $P_p = \mathrm{Id} - Q_p$. Note that $L_p(\mathcal{M}) \simeq L_p^\circ(\mathcal{M}) \oplus_p F_p$. When S_t are Fourier multipliers over $\mathcal{M} = \mathcal{L}G$ with symbol $e^{-t\psi}$ we define $G_0 = \{g \in G : \psi(g) = 0\}$. In that case

$$F_p = \overline{\Big\{ x \in L_p(\mathcal{M}) : \ x = \lambda(\nu) \text{ with } \mathrm{supp}(\nu) \subset G_0 \Big\}}$$

and in a similar way we find that $\lambda(\nu) \in L_p^\circ(\mathcal{M})$ if and only if $\mathrm{supp}\nu \subset G \setminus G_0$.

For any given $x \in \mathcal{M}$ we define the function $Tx : (0, \infty) \to L_p(\mathcal{M})$ given by $t \mapsto t\,\partial_t S_t x$. We can see $x \mapsto Tx$ as a map from certain domain $D \subset \mathcal{M}$ into $L_p(\mathcal{M}; H^r)$, $L_p(\mathcal{M}; H^c)$ or $L_p(\mathcal{M}; H^{rc})$, where $H = L_2(\mathbb{R}_+, dt/t)$. The induced seminorms on $D \subset \mathcal{M}$ are called the row Hardy space, column Hardy space or Hardy space seminorms. Observe that the map T has as kernel those elements fixed by $\mathcal{S}$. Quotient out the nullspace and taking the completion with respect to any of those norms when $p < \infty$ (resp. the weak-$*$ topology for $p = \infty$) gives the Hardy spaces $H_p^r(\mathcal{M}; \mathcal{S})$, $H_p^c(\mathcal{M}; \mathcal{S})$ or $H_p(\mathcal{M}; \mathcal{S})$. We can represent such norms as follows

$$\|x\|_{H_p^c(\mathcal{M};\mathcal{S})} \quad = \quad \Big\| \Big(\int_{\mathbb{R}_+} \Big(t \frac{d}{dt} S_t x \Big)^* \Big(t \frac{d}{dt} S_t x \Big) \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})},$$

$$\|x\|_{H_p^r(\mathcal{M};\mathcal{S})} \quad = \quad \Big\| \Big(\int_{\mathbb{R}_+} \Big(t \frac{d}{dt} S_t x \Big) \Big(t \frac{d}{dt} S_t x \Big)^* \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}.$$

We will drop the dependency on the semigroup and write $H_p^c(\mathcal{M})$ whenever it can be understood from the context. These spaces inherit their o.s.s. from that of $L_p(\mathcal{M}; H^r)$ or $L_p(\mathcal{M}; H^c)$. Therefore we have the following identities

$$\begin{aligned} S_p^n[H_p^c(\mathcal{M};\mathcal{S})] &= H_p^c(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2^n); \mathcal{S} \otimes \mathrm{Id}), \\ S_p^n[H_p^r(\mathcal{M};\mathcal{S})] &= H_p^r(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2^n); \mathcal{S} \otimes \mathrm{Id}). \end{aligned}$$

The duality is obtained from that of $L_p(\mathcal{M}; H^c)$ or $L_p(\mathcal{M}; H^r)$, resulting in the cb-isometries $H_p^r(\mathcal{M};\mathcal{S})^* = H_{p'}^r(\mathcal{M}_{\mathrm{op}};\mathcal{S})$ for $1 \leq p < \infty$. The same holds for the column case. Finally let us recall that by [JLMX06, Chapters 7 and 10] we have that if $1 < p < \infty$ then

$$H_p(\mathcal{M};\mathcal{S}) \simeq L_p^\circ(\mathcal{M}), \tag{1.9.1}$$

with the equivalence as operator spaces depending on the constant p. The result fails for $p = 1, \infty$ and $H_1(\mathcal{M};\mathcal{S})$ is smaller in general than $L_1^\circ(\mathcal{M})$. Observe that $t\partial_t S_t x = \eta(tA)x$ where $\eta(z) = ze^{-z}$. Due to the results in [JLMX06] we can change η by other analytic functions in certain class obtaining equivalent norms. We will say that a holomorphic function ρ defined over the sector $\Sigma_\theta = \{z \in \mathbb{C} : |\arg(z)| < \theta\}$ is in $H^\infty(\Sigma_\theta)$ iff it is bounded and we will say that it is in $\mathcal{H}_0^\infty(\Sigma_\theta) \subset \mathcal{H}^\infty(\Sigma_\theta)$ iff there is an $s > 0$ such that

$$|\rho(z)| \lesssim \frac{|z|^s}{(1+|z|)^{2s}}.$$

We will denote by $\mathcal{H}^\infty$ or $\mathcal{H}_0^\infty$ the spaces

$$\mathcal{H}^\infty \quad = \bigcap_{0<\theta<\pi/2} \mathcal{H}^\infty(\Sigma_\theta) \quad \text{and} \quad \mathcal{H}_0^\infty \quad = \bigcap_{0<\theta<\pi/2} \mathcal{H}_0^\infty(\Sigma_\theta).$$

If needed, we will equip these spaces with their natural inverse limit topologies. We have that for any $\rho \in \mathcal{H}_0^\infty$ the following holds

$$\begin{aligned} \|x\|_{H_p^c(\mathcal{M})} &\sim_{(p)} \Big\| \Big(\int_{\mathbb{R}_+} \big(\rho(tA)x\big)^* \rho(tA)x \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \\ \|x\|_{H_p^r(\mathcal{M})} &\sim_{(p)} \Big\| \Big(\int_{\mathbb{R}_+} \rho(tA)x \big(\rho(tA)x\big)^* \frac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}. \end{aligned} \tag{1.9.2}$$

The equivalence also holds after matrix amplifications. This type of identities also hold for wider classes of unbounded operators A satisfying certain resolvent estimates, see [JLMX06] for further details.

1.10 Sobolev Dimension

Let $(\mathcal{M}, \tau)$ be a noncommutative measure space and consider a Markov semigroup $\mathcal{S} = (S_t)_{t \geq 0}$ defined on it. Given a positive function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $1 \leq p < q \leq \infty$, we say that $\mathcal{S}$ satisfies the $\mathrm{R}_\Phi^{p,q}$ ultracontractivity property when

$$\big\| S_t : L_p(\mathcal{M}) \to L_q(\mathcal{M}) \big\| \quad \lesssim \quad \frac{1}{\Phi(\sqrt{t})^{\frac{1}{p} - \frac{1}{q}}} \quad \forall\, t > 0. \tag{$\mathrm{R}_\Phi^{p,q}$}$$

Similarly, $\mathcal{S}$ has the $\mathrm{CBR}_\Phi^{p,q}$ property when the above estimate holds for the c.b. norm of $S_t : L_p(\mathcal{M}) \to L_q(\mathcal{M})$. These inequalities have been extensively studied for commutative measure spaces [VSCC92, Chapter 1]. In the theory of Lie groups with an invariant Riemannian metric (equipped with the heat semigroup generated by the invariant Laplacian) ultracontractivity holds for the function $\Phi(t) = \mu(B_t(e))$ which assigns the volume of a ball for a given radius. Influenced by that, we will interpret the above-defined properties as a way of describing the "growth of the balls" in the noncommutative geometry determined by $\mathcal{S} = (S_t)_{t\geq 0}$. For that reason, we will work with *doubling functions* Φ. Doubling functions are increasing functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Phi(0) = 0$ and satisfying

$$\sup_{t>0} \left\{ \frac{\Phi(2t)}{\Phi(t)} \right\} < \infty.$$

The doubling condition for Φ is a natural requirement since metric measure spaces (Ω, μ, d) with $\Phi_x(t) = \mu(B_x(t))$ uniformly doubling in x constitute an adequate setting for performing harmonic analysis in commutative measure spaces. Given a Markov semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ over a noncommutative measure space $(\mathcal{M}, \tau)$, let us recall the following:

i) If $\mathcal{S}$ satisfies $\mathrm{R}_\Phi^{p_0,q_0}$, it satisfies $\mathrm{R}_\Phi^{p,q}$ for $1 \leq p_0 \leq p < q \leq q_0 \leq \infty$.

ii) If Φ is doubling and $\mathcal{S}$ satisfies $\mathrm{R}_\Phi^{p_0,q_0}$ for some $1 \leq p_0 < q_0 \leq \infty$, then it satisfies $\mathrm{R}_\Phi^{p,q}$ for $1 \leq p \leq q \leq \infty$

The same holds for the $\mathrm{CBR}_\Phi^{p_0,q_0}$ ultracontractivity property. The proof follows the same lines than [VSCC92, Theorem II.1.3]. In the noncommutative setting a similar result is stated in [JM10, Lemma 1.1.2] for $\Phi(t) = t^D$. As a consequence, all the ultracontractivity properties $\mathrm{R}_\Phi^{p,q}$ are equivalent for doubling Φ. We shall denote them simply by R_Φ and similarly CBR_Φ. As a corollary, we obtain that if $\mathcal{M}$ is an abelian von Neumann algebra $\mathrm{CBR}_\Phi^{p,q}$ and $\mathrm{R}_\Phi^{p,q}$ are equivalent for doubling Φ since $\mathrm{R}_\Phi^{p,q}$ is equivalent to $\mathrm{R}_\Phi^{p,\infty}$ and any bounded map into an abelian C^*-algebra is completely bounded. For any doubling function Φ we may define its doubling dimension D_Φ as

$$D_\Phi = \log_2 \sup_{t>0} \left\{ \frac{\Phi(2t)}{\Phi(t)} \right\}.$$

It is quite simple to show that any doubling $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ admits upper/lower polynomial bounds for large/small values of $t > 0$. More precisely, we have the bounds

$$\begin{array}{rcccl} \Phi(t) & \lesssim_{(D_\Phi)} & t^{D_\Phi}\, \Phi(1) & \text{when} & t > 1, \\ \Phi(t) & \gtrsim_{(D_\Phi)} & t^{D_\Phi}\, \Phi(1) & \text{when} & t \leq 1. \end{array} \tag{1.10.1}$$

Of course, the converse of this assertion is false. Whenever a Markovian semigroup $\mathcal{S}$ satisfies R_Φ (resp. CBR_Φ) for doubling Φ we will call D_Φ the *Sobolev dimension* (resp. *c.b. Sobolev dimension*) of $(\mathcal{M}, \tau)$ with respect to $\mathcal{S}$. The reason for this name is based on the well-known relation between ultracontractivity estimates for a Markov semigroup and Sobolev embedding estimates for its infinitesimal generator. One of the first contributions to that relation is in the work of Varopoulos, who proved in [Var85b] that when $\Phi(t) = t^D$ the property R_Φ is equivalent to a whole range of Sobolev type estimates for the infinitesimal generator of the semigroup. See also [VSCC92] for more on that topic. Whenever $\Phi(t) = t^D$ we will denote the ultracontractivity properties by R_D or CBR_D. By adding a zero, like $\mathrm{R}_\Phi(0)$, we will mean that the inequality $\mathrm{R}_\Phi^{p,q}$ is satisfied for $t \leq 1$. This notation is borrowed from [VSCC92, II.5]. Recall that if $\mathcal{S}$ satisfies R_Φ (resp. CBR_Φ) for some doubling function Φ then, by the polynomial bounds in (1.10.1), we have $\mathrm{R}_{D_\Phi}(0)$ (resp. $\mathrm{CBR}_{D_\Phi}(0)$).

We also recall that, as a consequence of the classification of c.b. multipliers $T_m : L_2(\mathcal{L}G) \to \mathcal{L}G$, the condition $\mathrm{CBR}_\Phi^{2,\infty}$ is quite restrictive. Indeed, it implies, not just the Haagerup property, but amenability of $\mathcal{L}G$.

Remark 1.10.1. As a consequence of the above, if G is a group and $\mathcal{S} = (T_{e^{-t\,\psi}})_{t\geq 0}$ is a semigroup of Fourier multipliers satisfying $\mathrm{CBR}_\Phi^{2,\infty}$ for any function Φ, then G is amenable. To see it just notice that $e^{-t\psi} \in L_2(G)$ and so $e^{-2t\psi} \in L_1(G)$ for all $t > 0$. But a group is amenable iff there is a sequence of integrable positive type functions converging to 1 uniformly in compacts.

Our characterization of co-polynomial growth in Chapter 2 requires the following equivalence for Sobolev-type inequalities in term of the ultracontractivity properties $\mathrm{R}_\Phi^{p;q}$. We did not find the proposition below in the literature, but it could be well-known to experts. We include a sketch of the proof.

Proposition 1.10.2. *Let $\mathcal{S}$ be a submarkovian semigroup acting on a noncommutative measure space $(\mathcal{M}, \tau)$. Let A denote its infinitesimal generator. Then, the following properties are equivalent:*

i) *For every $\varepsilon > 0$, $\mathcal{S}$ satisfies the $\mathrm{R}_{D+\varepsilon}(0)$ property.*

ii) *For every $\varepsilon > 0$, we have that*

$$\|(\mathbf{1} + A)^{-D/4-\varepsilon} : L_2(\mathcal{M}) \to \mathcal{M}\| \lesssim_{(\varepsilon)} 1.$$

Similarly, $\mathcal{S} \in \mathrm{CBR}_{D+\varepsilon}(0)$ for all $\varepsilon > 0$ iff $(\mathbf{1} + A)^{-s} : L_2(\mathcal{M}) \xrightarrow{\mathrm{cb}} \mathcal{M}$ for all $\varepsilon > 0$.

Proof. The implication i) $\Rightarrow$ ii) follows from the identity

$$(\mathbf{1} + A)^{-s}(x) \quad = \quad \frac{1}{\Gamma(s)} \Big(\int_{\mathbb{R}_+} t^s e^{-t} S_t(x) \frac{dt}{t} \Big).$$

The integral in $[0, 1]$ may be estimated applying the $\mathrm{R}_D(0)$ property, whereas the integral for $t > 1$ is easily estimated using the semigroup law. This gives the desired implication. For the converse, we now take $s = D/4 + \varepsilon$ and use that $\|f(A)\|_{\mathcal{B}(L_2)} \leq \|f\|_\infty$

$$\begin{aligned} \big\|S_t : L_2(\mathcal{M}) \to \mathcal{M}\big\| &= \big\|(\mathbf{1} + A)^{-\frac{s}{2}} (\mathbf{1} + A)^{\frac{s}{2}} S_t\big\|_{\mathcal{B}(L_2(\mathcal{M}),\mathcal{M})} \\ &\leq \big\|(\mathbf{1} + A)^{-\frac{s}{2}}\big\|_{\mathcal{B}(L_2(\mathcal{M}),\mathcal{M})} \big\|(\mathbf{1} + A)^{\frac{s}{2}} S_t\big\|_{\mathcal{B}(L_2)} \lesssim_{(\varepsilon,s)} \Big(\frac{s}{2}\Big)^{\left(\frac{s}{2}\right)} e^{-\frac{s}{2}} \frac{e^t}{t^{\frac{s}{2}}}. \end{aligned}$$

□

Remark 1.10.3. Observe that if $\mathrm{R}_D(0)$ is satisfied then ii) also holds. Nevertheless the converse is not true since the norm $\|S_t : L_1(\mathcal{M}) \to \mathcal{M}\|$ could be comparable to, say, $t^D(1 + \log(t))$ for $0 \leq t \leq 1$. The original result proved by Varopoulos [Var85b] established a equivalence between $\mathrm{R}_D(0)$ and the bounds

$$(\mathbf{1} + A)^{-s} : L_p(\mathcal{M}) \to L_{\frac{pn}{n-sp}}(\mathcal{M})$$

for every $0 \leq s < n/p$. When $s > n/p$ the image space of $L_p(\mathcal{M})$ is certainly much smaller than $L_\infty(\mathcal{M})$, for example in $\mathbb{R}^n$ with the usual Laplacian the image space lies inside spaces of Hölder functions. Therefore, by describing the behavior of $(\mathbf{1} + A)^{-s}$ in $L_\infty(\mathcal{M})$ we lose information and we can no longer recover $\mathrm{R}_D(0)$.

We will denote by $W_A^{p,s}(\mathcal{M})$, or simply $W^{p,s}(\mathcal{M})$ when the semigroup $S_t = e^{-tA}$ can be understood from the context, the closed domain in $L_p(\mathcal{M})$ of the unbounded operator $(\mathbf{1}+A)^{s/2}$, with norm given by

$$\|x\|_{W_A^{p,s}} = \big\|(\mathbf{1}+A)^{s/2} f\big\|_p.$$

These are called the *fractional Sobolev spaces* associated with $\mathcal{S}$. They satisfy the natural interpolation identities. Namely, if we set $1/p_3 = (1-\theta)/p_1 + \theta/p_2$ we get

$$\begin{aligned} \big[W_A^{p_1,s}(\mathcal{M}), W_A^{p_2,s}(\mathcal{M})\big]_\theta &\simeq W_A^{p_3,s}(\mathcal{M}), \\ \big[W_A^{p,s_1}(\mathcal{M}), W_A^{p,s_2}(\mathcal{M})\big]_\theta &\simeq W_A^{p,s_1\theta+s_2(1-\theta)}(\mathcal{M}), \end{aligned}$$

Point ii) in Proposition 1.10.2 may be rephrased as $W_A^{2,s}(\mathcal{M}) \subset \mathcal{M}$ for every $s > D/2$.

Chapter 2

Maximal Bounds and Multipliers

In this section we are going to prove a principle of boundedness of noncommutative Fourier multipliers by noncommutative maximal operators. That principle will be used later on in the current chapter, see Section 2.2, to obtain nonradial multipliers analogous to the Marcinkiewicz multiplier Theorem, see [Duo01].

Let us first put in context our maximal estimates for Fourier multipliers. Given a symbol $m : \mathbb{R}^n \to \mathbb{C}$ with corresponding Fourier multiplier T_m, there is a long tradition in identifying maximal operators $\mathcal{M}$ which satisfy the weighted L_2-norm inequality below for all admissible input functions f and weights w

$$\int_{\mathbb{R}^n} |T_m f|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 \mathcal{M} w. \tag{2.0.1}$$

Such results go back at least to the work of Córdoba and Fefferman in the 70's. This general principle has deep connections with Bochner-Riesz multipliers and also with A_p weight theory. The Introduction of [Ben14] gives a very nice historical summary and new results in this direction. The main purpose of this estimate is that elementary duality arguments yield for $p > 2$ that

$$\big\| T_m : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \big\| \lesssim \big\| \mathcal{M} : L_{(p/2)'}(\mathbb{R}^n) \to L_{(p/2)'}(\mathbb{R}^n) \big\|^{\frac{1}{2}}.$$

We extend such principle to the noncommutative case. Such enterprise requires the use of nontrivial ingredients already reviewed in the prerequisites section —like the theory of noncommutative maximal functions or the $\mathcal{H}^\infty$-functional calculus and semigroup type Hardy spaces—.

As an easy application of such principle we are going to prove a robust multiplier result on polynomial growth Lie groups, see Theorem 2.2.9. Indeed, if $\mathbb{X} = \{X_1, ..., X_r\} \subset T_e G$ is a system of right invariant vector fields generating the whole Lie algebra we obtain that, for every $s > D_0/2$, with D_0 being the local dimension of $\mathbb{X}$, that

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \max\Big\{ \sup_{t \geq 0} \|m\, \eta(t\,\psi)\|_{W^{2,s}_{\mathbb{X}}(G)}, \sup_{t \geq 0} \|(\tilde{m})\, \eta(t\,\psi)\|_{W^{2,s}_{\mathbb{X}}(G)} \Big\}, \quad \forall\, 1 < p < \infty,$$

where $\|f\|_{W^{2,s}(G)} = \|(\mathbf{1} + \Delta_{\mathbb{X}})^{s/2} f\|_2$ is the Sobolev norm associated with the sublaplacian $\Delta_{\mathbb{X}}$ associated with $\mathbb{X}$ and $\tilde{m}(g) = m(g^{-1})$. In fact, such result is a particular case of a more general theorem on unimodular, LCH groups endowed with Markovian processes whose generator satisfies a dimensional conditions, see Theorem 2.2.8. The principle of boundedness by a maximal function will by used one again the Chapter 3 to prove spectral Hörmander-Mikhlin theorems.

The content of this Chapter corresponds roughly to the Sections 1 and 3 of [GPJP15]

2.1 Maximal Bounds

Let us introduce the following notion of decomposition, that will be crucial in order to bound multipliers by maximal functions.

Definition 2.1.1. Let $(B_t)_{t\geqslant 0}$ be a family of operators affiliated to $\mathcal{L}G$. We say that $(B_t)_{t\geq 0}$ has an L_p-*square-max decomposition* when there is a decomposition $B_t = \Sigma_t \, M_t$ such that

$$\begin{array}{rcl} \sup_{t\geq 0} \|\Sigma_t\|_2 & < & \infty, \\ \Big\| \sup_{t>0}{}^+ \sigma |M_t|^2 \star u \Big\|_p & \lesssim_{(p)} & \|u\|_p. \end{array} \qquad (\mathrm{SM}_p)$$

Similarly, $(B_t)_{t\geq 0}$ has an L_p-*max-square decomposition* when $B_t = M_t \, \Sigma_t$ with

$$\begin{array}{rcl} \sup_{t\geq 0} \|\Sigma_t\|_2 & < & \infty, \\ \Big\| \sup_{t>0}{}^+ \sigma |M_t^*|^2 \star u \Big\|_p & \lesssim_{(p)} & \|u\|_p. \end{array} \qquad (\mathrm{MS}_p)$$

When we say that $(B_t)_{t\geq 0}$ has a max-square (resp. square-max) decomposition we mean that it has an L_p-max-square (resp. L_p-square-max) decomposition for every $1 < p < \infty$.

Theorem 2.1.1. *Let G be a LCH group equipped with a conditionally negative length $\psi : G \to \mathbb{R}_+$. Let $\mathcal{S} = (S_t)_{t\geq 0}$ be the convolution semigroup generated by ψ and pick any $\eta \in \mathcal{H}_0^\infty$. If $m : G \to \mathbb{C}$ is a bounded function satisfying that $B_t = \lambda(m\,\eta(t\psi))$ has an $L_{(p/2)'}$-square-max decomposition $B_t = \Sigma_t \, M_t$ for some $2 < p < \infty$, then $T_m : H_p^c(\mathcal{L}G) \to H_p^c(\mathcal{L}G)$ and*

$$\|T_m\|_{\mathcal{B}(H_p^c)} \lesssim_{(p)} \Big(\sup_{t\geq 0} \|\Sigma_t\|_2 \Big) \Big\| (R_t^c)_{t\geq 0} : L_{(p/2)'}(\mathcal{L}G) \to L_{(p/2)'}(\mathcal{L}G; L_\infty) \Big\|^{\frac{1}{2}}$$

where $R_t^c(x) = \sigma |M_t|^2 \star x$. Similarly, when $(B_t)_{t\geq 0}$ admits an $L_{(p/2)'}$-max-square decomposition $B_t = M_t \, \Sigma_t$ for some $2 < p < \infty$, we get $T_m : H_p^r(\mathcal{L}G) \to H_p^r(\mathcal{L}G)$ and the following estimate holds

$$\|T_m\|_{\mathcal{B}(H_p^r)} \lesssim_{(p)} \Big(\sup_{t\geq 0} \|\Sigma_t\|_2 \Big) \Big\| (R_t^r)_{t\geq 0} : L_{(p/2)'}(\mathcal{L}G) \to L_{(p/2)'}(\mathcal{L}G; L_\infty) \Big\|^{\frac{1}{2}}$$

where $R_t^r(x) = \sigma |M_t^*|^2 \star x$. By duality, similar identities also hold for $1 < p < 2$.

Corollary 2.1.2. *If G, ψ, η and m are as above and $B_t = \lambda(m\,\eta(t\psi))$ admits both a $L_{(p/2)'}$-max-square and a $L_{(p/2)'}$-square-max decomposition, then it turns out that $T_m : L_p^\circ(\mathcal{L}G) \to L_p^\circ(\mathcal{L}G)$ boundedly. Furthermore, if $m \equiv c$ in $G_0 = \{g \in G : \psi(g)\}$ then T_m is a bounded map on $L_p(\mathcal{L}G)$.*

Proof. (of Corollary 2.1.2.) The first assertion follows trivially from (1.9.1). For the second we use that $L_p^\circ(\mathcal{L}G)$ is a complemented subspace, and so, we have

$$\begin{array}{rcl} \|T_m x\|_p & \leq & \|P_p T_m x\|_p + \|Q_p T_m x\|_p \\ & = & \|T_m P_p x\|_{L_p^\circ(\mathcal{L}G)} + \|T_{m_{|G_0}} Q_p x\|_p \\ & \lesssim_{(p)} & \Big(\|T_m\|_{\mathcal{B}(L_p^\circ(\mathcal{L}G))} + c \Big) \|x\|_p \end{array}$$

and the results follows. □

Proof. (of Theorem 2.1.1). Assume that $B_t = \lambda(m\,\eta(t\,\psi))$ has an $L_{(p/2)'}$-square-max decomposition. Since $\rho(z) = \eta(z)\varrho(z)$ is an $\mathcal{H}_0^\infty$ function for every $\varrho \in \mathcal{H}_0^\infty$, we have the square function estimates in (1.9.2). Using that T_m commutes with the spectral calculus of A, the generator of $\mathcal{S}$, we obtain

$$\begin{aligned} \|T_m(x)\|_{H_p^c} &\sim_{(p)} \big\|\big(\eta(tA)\varrho(tA)T_m x\big)_{t\geq 0}\big\|_{L_p(\mathcal{L}G;L_2^c)} \\ &= \big\|\big(\eta(tA)T_m\varrho(tA)x\big)_{t\geq 0}\big\|_{L_p(\mathcal{L}G;L_2^c)} \\ &= \big\|\big(T_{m_t}(x_t)\big)_{t\geq 0}\big\|_{L_p(\mathcal{L}G;L_2^c)}, \end{aligned}$$

where $m_t(g) = m(g)\,\eta(t\,\psi(g))$ and $x_t = T_{\varrho(t\psi)}x$. Recall also that the L_2-space involved is $L_2(\mathbb{R}_+, dt/t)$. Now we may express the term on the right hand side as follows

$$\begin{aligned} \big\|\big(T_{m_t}(x_t)\big)_{t\geq 0}\big\|^2_{L_p(\mathcal{L}G;L_2^c)} &= \Big\|\int_{\mathbb{R}_+} |T_{m_t}x_t|^2 \frac{dt}{t}\Big\|_{\frac{p}{2}} \\ &= \tau\Big\{u\int_{\mathbb{R}_+} |T_{m_t}x_t|^2 \frac{dt}{t}\Big\} \\ &= \int_{\mathbb{R}_+} \tau\big\{u|T_{m_t}x_t|^2\big\}\frac{dt}{t}, \end{aligned} \tag{2.1.1}$$

where $u \in L_{(p/2)'}(\mathcal{L}G)_+$ is the unique element realizing the $L_{p/2}$-norm, which exists by the weak-$*$ compactness of the unit ball of $L_{(p/2)'}(\mathcal{L}G)$. Now we have to estimate the term inside the integral. Since u is positive, we may write $u = w^*w$ for some $w \in L_{2(p/2)'}$ and

$$\begin{aligned} \big\langle u, |T_{m_t}(x_t)|^2\big\rangle &= \tau\{w\,|T_{m_t}(x_t)|^2 w^*\} \\ &= \tau\Big\{w\,\underbrace{\big|(\tau\otimes\mathrm{Id})\big(\delta B_t(\sigma x_t\otimes \mathbf{1})\big)\big|^2}_{L_t}\,w^*\Big\}. \end{aligned}$$

The map $L_t \mapsto wL_tw^*$ is order preserving, and so, any bound of L_t gives a bound of the term above. By the complete positivity of the canonical trace we can apply Proposition 1.1 in [Lan95], i.e.

$$\langle x,y\rangle^*\,\langle x,y\rangle \leq \|\langle x,x\rangle\|\langle y,y\rangle$$

to the operator-valued inner product $\langle x,y\rangle = (\tau\otimes\mathrm{Id})(x^*y)$. This yields

$$\begin{aligned} L_t &= \big|(\tau\otimes\mathrm{Id})\big\{\delta\Sigma_t\delta M_t(\sigma x_t\otimes\mathbf{1})\big\}\big|^2 \\ &\leq \big\|(\tau\otimes\mathrm{Id})\{\delta|\Sigma_t|^2\}\big\|_{\mathcal{L}G}\,(\tau\otimes\mathrm{Id})\big\{(\sigma x_t^*\otimes\mathbf{1})\delta M_t^*\delta M_t(\sigma x_t\otimes\mathbf{1})\big\} \\ &\leq \Big(\sup_{t>0}\|\Sigma_t\|_2^2\Big)\,(\tau\otimes\mathrm{Id})\big\{\delta|M_t|^2(\sigma(x_t^*x_t)\otimes\mathbf{1})\big\} \\ &= \Big(\sup_{t>0}\|\Sigma_t\|_2^2\Big)\,\big(|M_t|^2\star x_t^*x_t\big). \end{aligned}$$

We have used the δ-invariance of the trace in the second inequality and the definition of the noncommutative convolution in the last identity. Now, substituting inside the trace and using the identity for the adjoint of the noncommutative convolution operator gives

$$\begin{aligned} \big\langle u, |T_{m_t}(x_t)|^2\big\rangle &\leq K^2\,\tau\big\{u\,(|M_t|^2\star x_t^*x_t)\big\} \\ &= K^2\,\tau\big\{(\sigma|M_t|^2\star u)\,x_t^*x_t\big\}, \end{aligned}$$

where K is the supremum of the L_2 norm of Σ_t. This gives

$$\begin{aligned}
\|T_m(x)\|_{H_p^c}^2 &\lesssim_{(p)} K^2 \int_{\mathbb{R}_+} \tau\big\{(\sigma|M_t|^2 \star u) x_t^* x_t\big\} \frac{dt}{t} \\
&\leq K^2 \inf_{\sigma|M_t|^2 \star u \leq A} \tau\Big\{A \int_{\mathbb{R}_+} x_t^* x_t \frac{dt}{t}\Big\} \\
&\leq K^2 \inf_{\sigma|M_t|^2 \star u \leq A} \|A\|_{L_{(p/2)'}} \Big\| \int_{\mathbb{R}_+} x_t^* x_t \frac{dt}{t} \Big\|_{p/2} \\
&\lesssim_{(p)} K^2 \big\|(R_t^c)_{t\geq 0} : L_{(p/2)'} \to L_{(p/2)'}(L_\infty)\big\| \, \|x\|_{H_p^c}^2,
\end{aligned}$$

by using Fatou's lemma in the second line and the definition of the $L_p(\mathcal{L}G; L_\infty)$ norm for positive elements in the last inequality. Taking square roots gives the desired estimate. The calculations for the row case are entirely analogous. □

Remark 2.1.3. Throughout this paper we construct max-square and square-max decompositions of $B_t = \lambda(m\,\eta(t\,\psi))$ by choosing an smoothing positive factor M_t with $M_t = \sigma M_t = M_t^*$ and satisfying the appropriate maximal inequalities. Then we extract M_t from the left and from the right of B_t as

$$\begin{aligned}
B_t &= (B_t\, M_t^{-1})\, M_t, \\
B_t &= M_t\, (M_t^{-1}\, B_t).
\end{aligned}$$

If the family $\Sigma_t = B_t\, M_t^{-1}$ is uniformly bounded in L_2 and B_t is self-adjoint, then, $M_t^{-1}\, B_t$ is also automatically uniformly in L_2 by the traciality of τ. Most of the times it will be enough to check one of the two decompositions.

Remark 2.1.4. The technique employed here gives complete bounds assuming that the maximal inequalities are satisfied with complete bounds. In order to prove that assertion, let us express the matrix extension $(T_m \otimes \mathrm{Id}_{M_n})$ as a matrix-valued multiplier whose symbol takes diagonal values. Indeed

$$(T_m \otimes \mathrm{Id}_{M_n})([x_{ij}]) = (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}_{M_n})\Big\{ \underbrace{\big(\delta\lambda(m) \otimes \mathbf{1}_{M_n}\big)}_{K} \big(\mathbf{1} \otimes [\sigma x_{ij}]\big)\Big\},$$

where K is the corresponding kernel affiliated with $\mathcal{L}G \,\overline{\otimes}\, \mathcal{L}G \,\overline{\otimes}\, \mathbb{C}\mathbf{1}_{M_n}$. Clearly, any square-max decomposition $B_t = \Sigma_t M_t$ of $B_t = \lambda(m\,\eta(t\psi))$ yields a diagonal decomposition $(\delta\Sigma_t \otimes \mathbf{1}_{M_n})(\delta M_t \otimes \mathbf{1}_{M_n})$ of $K_t = \delta B_t \otimes \mathbf{1}_{M_n}$. On the other hand recall that $T_m : H_p^c \to H_p^c$ is c.b. iff $T_m \otimes \mathrm{Id}_{M_n} : S_p^n[H_p^c] \to S_p^n[H_p^c]$ is uniformly bounded for $n \geq 1$ and that $S_p^n[H_p^c(\mathcal{L}G; \mathcal{S})] = H_p^c(M_n \otimes \mathcal{L}G; \mathrm{Id} \otimes \mathcal{S})$. That allows us to write the norm of $S_p^n[H_p^c(\mathcal{L}G; \mathcal{S})]$ as an $L_{p/2}$-norm like in (2.1.1). Then, using [Lan95, Proposition 1.1] for $\langle x, y\rangle = (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}_{M_n})(x^* y)$ as in the proof of Theorem 2.1.1, gives for $2 < p < \infty$

$$\|T_m\|_{\mathcal{CB}(H_p^c)} \lesssim_{(p)} \Big(\sup_{t \geq 0} \|\Sigma_t\|_2\Big) \big\|(R_t^c)_{t\geq 0} : L_{(p/2)'}(\mathcal{L}G) \to L_{(p/2)'}(\mathcal{L}G; L_\infty)\big\|_{\mathrm{cb}}^{\frac{1}{2}}.$$

The row case is similar. The discussion of Corollary 2.1.2 generalizes to c.b. norms, and so, by interpolation, the formula above holds for the $\mathcal{CB}(L_p(\mathcal{L}G))$-norm of T_m provided there are simultaneously max-square and square-max inequalities.

2.2 Nonradial multipliers

In this section we will prove some smooth multiplier theorems applying the principle of bondedness by a maximal function. In order to formulate a smooth multiplier theorem we need a way to measure smoothness in general LCH groups. For that end we will choose to use the generators of, right and left translation invariant subarkovian semigroups over $L_\infty(G)$. The theorem will require a Sobolev-type inequality for such semigroups. Such inequality, after identifying the generator over G with its multiplication symbol in $\mathcal{L}G$ will be equivalent to an asymptotic condition, that we have called co-polynomial growth, that can be thought of as a noncommutative generalization of polynomial growth. In subsection 2.2.2 we study the particular case of polynomial growth Lie groups.

2.2.1 Polynomial co-growth

As we have seen, elements in the extended positive cone $\mathcal{L}G_+^\wedge$ can be understood as quantized metrics over $\mathcal{L}G$. Indeed, when G is abelian, any invariant distance over its dual group is determined by the (positive unbounded) function $d(e,\chi)$ affiliated to $L_\infty(\widehat{G})$, since $d(\chi_1,\chi_2) = d(e,\chi_1^{-1}\chi_2)$. It may seem natural to require X to satisfy properties analogous to the triangular inequality, the faithfulness and the symmetry. Nevertheless, such assumptions will not be necessary here since we will need just "asymptotic" properties of X. Indeed, one of our main families of examples will come from the unbounded multiplication symbols of invariant Laplacians over G. In order to match the classical case of $\mathbb{R}^n$ with the standard Laplacian, whose multiplication symbol is $|\xi|^2$, we will use the convention that X behaves like $d(e,\chi)^2$. That will explain the $1/2$ exponent in some of the formulas.

Definition 2.2.1. Given $X \in \mathcal{L}G_+^\wedge$, we say that X has polynomial co-growth of order D iff

$$D = \inf \Big\{ r > 0 : \big(\mathbf{1} + X\big)^{-r/2} \in L_1(\mathcal{L}G) \Big\} < \infty.$$

The definition is motivated by the fact that if we are in an abelian group and X is the unbounded positive function given by $d(e,\chi)^2$, where d is a translation invariant metric then, defining $\Phi(r) = \tau(\chi_{[0,r^2)}(X)) = \mu(B_r(e))$, we get

$$\begin{aligned} \big\| \big(\mathbf{1}+X\big)^{-D/2-\varepsilon} \big\|_1 &= \int_{\mathbb{R}_+} \frac{1}{(1+r^2)^{\frac{D}{2}+\varepsilon}} \, d\Phi(r) \\ &= \Big(\frac{D}{2}+\varepsilon\Big) \int_{\mathbb{R}_+} \frac{2\,r\,\Phi(r)}{(1+r^2)^{\frac{D}{2}+1+\varepsilon}} \, d\,r. \end{aligned}$$

In particular the last expression is finite whenever $\mu(B_r(e)) \lesssim r^D$.

Remark 2.2.1. In the proof of Theorem 2.2.8 we are only going to use that the convolution operator $u \mapsto u \star (1+X)^{-\beta}$ is completely bounded on $L_p(\mathcal{L}G)$ for $\beta > D$. Any element in $L_1(\mathcal{L}G)$ induces such bounded operator. Indeed we could have defined a similar notion of polynomial co-growth alternatively as

$$D = \inf \Big\{ r > 0 : (1+X)^{-r/2} \in \mathcal{CB}(L_1(\mathcal{L}G)) \cong M_{\mathrm{cb}}(AG) \Big\} < \infty,$$

where $(\mathbf{1}+X)^{-r/2}$ is identified with the operator $x \mapsto (\mathbf{1}+X)^{-r/2} \star x$. Observe that, since $\lambda : AG \to L_1(\mathcal{L}G)$ is an isomorphism, $\mathcal{CB}(L_1(\mathcal{L}G))$ is the more familiar algebra of c.b. multipliers over AG, the Fourier-Stjelties algebra This condition is a priori weaker than co-polynomial growth. We will stick to the original since it is a condition general enough to allow us to prove Theorem 2.2.8 and restrictive enough to be fully characterized.

Now we are going to prove the existence of unbounded operators affiliated to $\mathcal{L}G$ behaving like multiplication symbols for left or right invariant Laplacians. Recall that a submarkovian semigroup $\mathcal{S}$ acting on $L_\infty(G)$ is respectively called left/right invariant when $S_t \circ \lambda_g = \lambda_g \circ S_t$ or $S_t \circ \rho_g = \rho_g \circ S_t$ accordingly.

Proposition 2.2.2. *Let G be a LCH unimodular group and consider any submarkovian semigroup $\mathcal{S}$ over $L_\infty(G)$. Let A denote its positive generator. Then, the following properties hold:*

i) *If $\mathcal{S}$ is left invariant then there is a densely defined and closable unbounded positive operator $\widehat{A}$ affiliated to $\mathcal{L}G$ such that, for all $f \in \mathrm{dom}(A) \subset L_2(G)$*

$$\lambda(Af) = \lambda(f)\widehat{A}.$$

ii) *If $\mathcal{S}$ is right invariant then there is densely defined and closable unbounded positive operator $\widehat{A}$ affiliated to $\mathcal{L}G$ such that, for all $f \in \mathrm{dom}(A) \subset L_2(G)$*

$$\lambda(Af) = \widehat{A}\lambda(f).$$

Proof. We start by proving ii). Notice that $A : \mathrm{dom}(A) \subset L_2(G) \to L_2(G)$ is densely defined. It is affiliated with $\mathcal{L}G$ iff for every unitary $u \in \mathcal{L}G' = \mathcal{R}G$ we have that $uA = Au$. Since S_t is ρ invariant and we can approximate in the SOT topology every element in $\mathcal{R}G$ by linear combinations of elements in $(\rho_g)_{g \in G}$, we obtain that S_t commutes with any element $x \in \mathcal{R}G$. A function $f \in L_2(G)$ is in $\mathrm{dom}(A)$ when

$$\lim_{t \to 0^+} \frac{\mathrm{Id} - S_t}{t} f$$

exists in $L_2(G)$ and we then have

$$\lim_{t \to 0^+} \Big\| Af - \frac{\mathrm{Id} - S_t}{t} f \Big\|_2 = 0.$$

This implies $u\,\mathrm{dom}(A) \subset \mathrm{dom}(A)$ for any $U(\mathcal{R}G)$. Multiplying by u we obtain

$$\big\| uAf - Auf \big\|_2 \leq \lim_{t \to 0^+} \Big\| uAf - \frac{\mathrm{Id} - S_t}{t} uf \Big\|_2 + \lim_{t \to 0^+} \Big\| \frac{\mathrm{Id} - S_t}{t} uf - Auf \Big\|_2 = 0$$

for every $f \in \mathrm{dom}(A)$. This proves that A is affiliated with $\mathcal{R}G$. Notice that $\lambda : L_2(G) \to L_2(\mathcal{L}G)$ unitarily. We will define $\widehat{A} = \lambda A \lambda^*$. By definition $\widehat{A}$ is an unbounded operator on $L_2(\mathcal{L}G)$ affiliated with $(\lambda \mathcal{R}G \lambda^*)' = \lambda \mathcal{L}G \lambda^*$ which is also equal to the von Neumann algebra $\mathcal{L}G$ acting by left multiplication in the GNS construction associated to its trace. The operator $\widehat{A}$ is densely defined and closable since A is densely defined and closable. The identity of ii) follows by definition. The construction for i) is somewhat analogous. We need two trivial observations:

1. The anti-automorphism $\sigma : \mathcal{L}G \to \mathcal{L}G$ extends to a unitary operator $\sigma_2 : L_2(\mathcal{L}G) \to L_2(\mathcal{L}G)$ since $\tau \circ \sigma = \tau$. If $\pi_r : \mathcal{L}G_{\mathrm{op}} \to \mathcal{B}(L_2(\mathcal{L}G))$ and $\pi_\ell : \mathcal{L}G \to \mathcal{B}(L_2(\mathcal{L}G))$ are the right and left GNS representations, then $\sigma_2 \circ \pi_r(x) = \pi_\ell(\sigma x) \circ \sigma_2$.

2. The anti-automorphism σ extends to an automorphism of the extended positive cone $\mathcal{L}G_+^\wedge$. We are going to denote such extension again by σ.

Notice that, since $\pi_\ell[\mathcal{L}G]' = \pi_r[\mathcal{L}G]$, any element in $x \in \pi_\ell[\mathcal{L}G]'$ can be expressed as $\pi_r(x')$ for some $x' \in \mathcal{L}G$. By point 1, the map that sends x to x' is given, after identifying $\mathcal{L}G$ with its GNS representation $\pi_\ell[\mathcal{L}G]$, by $x' = \sigma(\sigma_2\, x\, \sigma_2)$. Let S be given by $S = \lambda A \lambda^*$. Then S is affiliated with $(\lambda \mathcal{L}G \lambda^*)' = \pi_\ell[\mathcal{L}G]'$. If we define $\widehat{A}$ as $\widehat{A} = \sigma(\sigma_2\, S\, \sigma_2)$, where σ is the extension of point 2, we obtain i). □

Remark 2.2.3. Since G is unimodular, the unitary $\iota : L_2(G) \to L_2(G)$ given by $f(g) \mapsto f(g^{-1})$ is an isometry that intertwines ρ_g and λ_g. We can characterize the pairs of self-adjoint left and right invariant operators A_1, A_2 whose left and right multiplication symbols, $\widehat{A}_1$ and $\widehat{A}_2$ respectively, coincide. By a trivial calculation those are the operators such that $A_1 \iota = \iota A_2$. Indeed, using that $\lambda : L_2(G) \to L_2(\mathcal{L}G)$ satisfies $\lambda \circ \iota = \sigma_2 \circ \lambda$ and that if A is the infinitesimal generator of a submarkovian semigroup then $A^\top = A$, we obtain that

$$\sigma(\lambda A_1 \lambda^*) = \sigma_2 \lambda A_2 \lambda^* \sigma_2 = \lambda \iota A_2 \iota \lambda^*,$$

but the right hand side satisfies that $\sigma(\lambda A_1 \lambda^*) = \lambda A_1^\top \lambda^* = \lambda A_1 \lambda^*$.

Now we are going to characterize those semigroups whose infinitesimal generator has polynomial co-growth. In order to prove the characterization we will need the following two lemmas. Recall that the Fourier algebra AG is defined as those $f : G \to \mathbb{C}$ such that $\lambda(f) \in L_1(\mathcal{L}G)$ with $\|f\|_{AG} = \|\lambda(f)\|_{L_1(\mathcal{L}G)}$. We will use below the straightforward inequalities for $f \in AG$

$$|\tau(\lambda(f))| \leq \|f\|_\infty \leq \tau(|\lambda(f)|). \tag{2.2.1}$$

Indeed, both follow from the identity $\tau(\lambda_g^* \lambda(f)) = f(g)$ which is valid for $f \in AG$.

Lemma 2.2.4. *Let G be a LCH unimodular group and $\mathcal{S}$ a submarkovian semigroup of right, resp. left, invariant operators satisfying that $S_t : C_0(G) \to C_0(G)$. Let A be the positive generator and assume further that $\widehat{A}$ has polynomial cogrowth of order D. Then $W_A^{2,s}(G) \cap AG$ is dense inside $W_A^{2,s}(G)$ for every $s > D/2 + \varepsilon$.*

Proof. We will prove only the right invariant case. Notice that AG is closed by left and right translations. The fact that $S_t : C_0(G) \to C_0(G)$, together with the Riesz representation theorem gives that for every $g \in G$ there is weak-$*$ continuous family of unit measures on G, $(\mu_t^g)_{g \in G, t \geq 0}$ such that

$$S_t f(g) = \int_G f(h)\, d\,\mu_t^g(h).$$

Applying the right invariance gives us that $d\mu_t^g(h) = d\mu_t^e(h\, g^{-1})$. This yields

$$S_t f(g) = \int_G \rho_g f\, d\,\mu_t^e = \iota^* \mu_t^e * f(g), \tag{2.2.2}$$

where $(\iota^* \mu_t^e)(E) = \mu_t^e(E^{-1})$. It is clear that $\|S_t f - f\|_{L_2(G)} \to 0$ as $t \to 0^+$. Recall that the same is true for $f \in W_A^{2,s}(G)$ in the $W_A^{2,s}(G)$-norm for every $s > 0$. Suppose that $f \in W_A^{2,s}(G)$, then, applying the formula (2.2.2) together with the polynomial co-growth, we have that

$$S_t f = \iota^* \mu_t * f = \iota^* \mu_t * (\mathbf{1} + A)^{-\frac{s}{2}} (\mathbf{1} + A)^{\frac{s}{2}} f = h_{t,s} * g,$$

where $g = (\mathbf{1} + A)^{s/2} f$. We have that $\|g\|_2 = \|f\|_{W_A^{s,2}}$ and

$$\|h_{t,s}\|_2 \leq \big\|\iota^* \mu_t * (\mathbf{1} + \widehat{A})^{-s/2}\big\|_{L_2(\mathcal{L}G)} \leq \|\mu_t^e\| \, \big\|(\mathbf{1} + \widehat{A})^{-s/2}\big\|_{L_2(\mathcal{L}G)} < \infty.$$

This proves that $S_t f \in AG \cap W_A^{2,s}(G)$. Making $t \to 0^+$ completes the claim. □

Theorem 2.2.5. *Let G be a unimodular LCH group and let $\mathcal{S}$ be a right (resp. left) invariant submarkovian semigroup over G. Let A be its infinitesimal generator and assume further that $S_t : C_0(G) \to C_0(G)$. Then, the following assertions are equivalent:*

(i) *The multiplication symbol $\widehat{A}$ of A has polynomial co-growth of order D.*

(ii) *$\mathcal{S}$ satisfies the following inequality for every $\varepsilon > 0$*

$$\Big\| (\mathbf{1}+A)^{-(\frac{D}{4}+\varepsilon)} : L_2(G) \to L_\infty(G) \Big\| \lesssim_{(\varepsilon)} 1.$$

Proof. To prove (i) $\Rightarrow$ (ii), pick $f \in AG \cap W^{2,s}(G)$ for $s = D/2 + 2\varepsilon$ and note

$$\begin{aligned}
\|f\|_\infty &\leq \big\|\lambda\big((\mathbf{1}+A)^{-s/2}(\mathbf{1}+A)^{s/2} f\big)\big\|_1 \\
&= \big\|\big(\mathbf{1}+\widehat{A}\big)^{-s/2}\lambda\big((\mathbf{1}+A)^{s/2} f\big)\big\|_1 \\
&\leq \big\|\big(\mathbf{1}+\widehat{A}\big)^{-s/2}\big\|_2 \big\|\lambda\big((\mathbf{1}+A)^{s/2} f\big)\big\|_2 \\
&= \big\|(\mathbf{1}+\widehat{A})^{-s}\big\|_1^{1/2} \|f\|_{W_A^{2,s}(G)} \lesssim_{(\varepsilon)} \|f\|_{W_A^{2,s}(G)}.
\end{aligned}$$

We have used (2.2.1) in the first inequality, Proposition 2.2.2 in the first identity and the polynomial cogrowth in the last inequality. By the density Lemma 2.2.4 we conclude that $W_A^{2,s}(G)$ embeds in $L_\infty(G)$ which is a rephrasal of ii). For the implication (ii) $\Rightarrow$ (i) we note that from (2.2.1)

$$\Big|\tau\Big(\big(\mathbf{1}+\widehat{A}\big)^{-\frac{D}{4}-\varepsilon}\lambda(f)\Big)\Big| \leq \big\|\big(\mathbf{1}+A\big)^{-\frac{D}{4}-2\varepsilon} f\big\|_\infty \lesssim_{(\varepsilon)} \|f\|_2.$$

Taking the supremum over $f \in L_2(G)$ with norm 1 gives the desired result. $\square$

Remark 2.2.6. Due to Proposition 1.10.2 we obtain that the point ii) is equivalent to satisfying the ultracontractivity property $\mathrm{R}_{D+\varepsilon}(0)$ for every $\varepsilon > 0$. Since $\mathrm{R}_D(0)$ implies $\mathrm{R}_{D+\varepsilon}(0)$ for every $\varepsilon > 0$, it is sufficient to prove $\mathrm{R}_D(0)$ in order to have polynomial co-growth of order D.

Remark 2.2.7. Sobolev inequalities involving powers of $\mathbf{1}+A$ are sometimes called local [VSCC92, II.X] since they are tightly connected to the ultracontractivity estimates for $0 < t \leq 1$ and in many contexts that amounts to describing the growth of ball of small radius. Therefore Theorem 2.2.5 relates the behaviour of the large balls of $\mathcal{L}G$ with the behaviour of small balls in G. This goes along the common intuition that taking group duals exchanges local and asymptotic/coarse properties.

Theorem 2.2.8. *Let G be a unimodular group equipped with a conditionally negative length ψ. Let $\mathcal{S}_1/\mathcal{S}_2$ be respectively left/right invariant submarkovian semigroups on $L_\infty(G)$ whose generators A_j satisfy* $\mathrm{cogrowth}(\widehat{A}_j) = D_j$ *for $j = 1, 2$. Consider an $\mathcal{H}_0^\infty$-cut-off function η and a symbol $m : G \to \mathbb{C}$ which is constant in the subgroup $G_0 = \{g \in G : \psi(g) = 0\}$. Then, if $s_j > D_j/2$ for $j = 1, 2$, the following inequality holds for $1 < p < \infty$*

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \max\Big\{ \sup_{t \geq 0} \big\|\eta(t\psi)\, m\big\|_{W_{A_1}^{2,s_1}(G)}, \sup_{t\geq 0} \big\|\eta(t\psi)\, m\big\|_{W_{A_2}^{2,s_2}(G)} \Big\}.$$

Proof. Let $B_t = \lambda(m\,\eta(t\psi))$ and let $\widehat{A}_1$ be the multiplication symbol associated with the generator of the right invariant semigroup $\mathcal{S}_1$ which is determined by Proposition 2.2.2. Then

$$B_t = \underbrace{\big(\mathbf{1}+\widehat{A}_1\big)^{-\frac{s_1}{2}}}_{M_t} \underbrace{\big(\mathbf{1}+\widehat{A}_1\big)^{\frac{s_1}{2}} B_t}_{\Sigma_t}$$

is a max-square decomposition. By the definition of co-polynomial growth we have that $\sigma|M_t|^2 = (\mathbf{1}+\sigma\widehat{A}_1)^{-s_1} \in L_1(\mathcal{L}G)$ and therefore it is a c.b. multiplier in every $L_p(\mathcal{L}G)$ for $1 \leq p \leq \infty$. Since M_t does not depend on t, the maximal inequality (MS_p) is satisfied trivially. By the construction of $\widehat{A}_1$ we have

$$\sup_{t>0} \|\Sigma_t\|_{L_2(\mathcal{L}G)} = \sup_{t>0} \big\| \big(\mathbf{1}+\widehat{A}_1\big)^{\frac{s_1}{2}} \lambda(m\,\eta(t\psi)) \big\|_{L_2(\mathcal{L}G)} = \sup_{t>0} \|m\,\eta(t\psi)\|_{W^{2,s_1}_{A_1}(G)}.$$

The square-max decomposition is manufactured in exactly the same way. □

2.2.2 Sublaplacians over polynomial-growth Lie groups

Here we are going to work with left (resp. right) invariant submarkovian semigroups over $L_\infty(G)$ generated by sublaplacians. Let M be a smooth manifold, $\mathbb{X} = \{X_1, .., X_r\}$ be a family of smooth vector fields and μ a σ-finite measure over M. Let us denote by $(\sigma_j(t))_{t\in(-\varepsilon_j,\varepsilon)_j}$ the one-parameter diffeomorphism generated by X_j and assume further that μ is invariant under $(\sigma_j(t))_{t\in(-\varepsilon_j,\varepsilon_j)}$. Then, the semigroup whose infinitesimal generator is given by the sublaplacian associated to $\mathbb{X}$

$$\Delta_{\mathbb{X}} = -\sum_{j=1}^{r} X_j^2$$

is submarkovian. This is a consequence of the theory of symmetric Dirichlet forms [FOT10]. If $M = G$ is a Lie group, μ its left Haar measure and $\mathbb{X} = \{X_1, ..., X_r\}$ left invariant vector fields. By the invariance under the one parameter subgroup generated by X_j of μ we have that $S_t = e^{-t\Delta_{\mathbb{X}}}$ is a submarkovian semigroup of left invariant operators. The same construction can be performed using right invariant vector fields if G is unimodular. Any sublaplacian carries a natural subriemannian metric given by

$$d_{\mathbb{X}}(x,y) = \inf_{\substack{\gamma:[0,1]\to M,\\ \gamma(0)=x,\gamma(1)=y}} \Big\{ \Big(\int_0^1 |\gamma'(t)|^2\,dt \Big)^{\frac{1}{2}} \mid \gamma'(t) \in \mathrm{span}\mathbb{X}_{\gamma(t)} \Big\}.$$

This metric coincides with the Lipschitz distance given by the gradient form, also known as Meyer's carré du champs [Mey76]. Observe also that, if G is a connected Lie group, then its subriemannian distance is finite iff $\mathbb{X}$ generates the whole Lie algebra. Similarly, $f \in \ker_p(\Delta_{\mathbb{X}})$ iff $f \in L_p(M)$ and $f(x) = f(y)$ whenever the subriemannian distance $d_{\mathbb{X}}(x,y)$ is finite.

The main family of illustrations of Theorem 2.2.8 comes from Lie groups endowed with right and left invariant sublaplacians. Let $\mathbb{V} = \{v_1, v_2, ..., v_r\} \subset T_eG$ be a collection of, linearly independent, vectors generating the whole Lie algebra, such sets are called *Hörmander system.* Let $\mathbb{X}_1 = \{X_1, ..., X_r\}$ and $\mathbb{X}_2 = \{Y_1, ..., Y_r\}$ be its right and left invariant extensions respectively. Then their associated sublaplacians satisfy $\iota\Delta_{\mathbb{X}_1} = \Delta_{\mathbb{X}_2}\iota$ where we use $\iota f(g) = f(g^{-1})$. Hence, it suffices to study the polynomial co-growth for $\widehat{\Delta}_{\mathbb{X}_1}$. By Remark 2.2.6 we just need to show that $S_t = e^{-t\Delta_{\mathbb{X}}}$ has the $\mathrm{R}_D(0)$ property and by [VSCC92, Theorem VIII.2.9] we known that if G is a Lie group of polynomial growth, then

$$\frac{e^{-\beta_1 \frac{d_{\mathbb{X}_1}(x,y)^2}{t}}}{\mu(B_e(\sqrt{t}))} \lesssim h_t(x,y) \lesssim \frac{e^{-\beta_2 \frac{d_{\mathbb{X}_1}(x,y)^2}{t}}}{\mu(B_e(\sqrt{t}))},$$

where h_t is the heat kernel associated with S_t, $d_{\mathbb{X}_1}$ is the subriemannian distance associated to $\mathbb{X}_1$ and $B_e(r)$ are the balls of radius r with respect to that metric. It is a well known fact, see

[VSCC92], that

$$\mu(B_e(r)) \sim t^{D_0},$$

for t small. Here D_0 is the *local dimension* associated to $\mathbb{X}_1$, given by

$$D_0 = \sum_{j=0}^{\infty} j \dim(F_{j+1}/F_j),$$

where $F_0 = \{0\}$, $F_1 = \mathbb{X}_1$ and $F_{j+1} = \mathrm{span}\{F_j, [F_j, \mathbb{X}_1]\}$. As a consequence S_t has the $R_{D_0}(0)$ property and therefore $\widehat{\Delta}_{\mathbb{X}_1}$, and so $\widehat{\Delta}_{\mathbb{X}_2}$, have polynomial co-growth of order D_0. As a corollary we obtain the following theorem.

Theorem 2.2.9. *Let G be a polynomial growth Lie group equipped with a c.n. length $\psi : G \to \mathbb{R}_+$ and let $\eta \in \mathcal{H}_0^\infty$, $\mathbb{X} = \{X_1, X_2, ...X_r\}$ be a Hörmander system and $\Delta_{\mathbb{X}}$ its sublaplacian. Then, for any $s > D_0/2$*

$$\|T_m\|_{\mathcal{CB}(L_p^\circ(\mathcal{L}G))} \lesssim_{(p)} \max\Big\{ \sup_{t\geq 0} \big\|\eta(t\psi)m\big\|_{W^{2,s}_{\Delta_{\mathbb{X}}}(G)}, \sup_{t\geq 0} \big\|\eta(t\psi)\iota m\big\|_{W^{2,s}_{\Delta_{\mathbb{X}}}(G)} \Big\},$$

for every $1 < p < \infty$.

Chapter 3

Hörmander-Mikhlin multipliers

The purpose of this chapter is to use the principle of boundedness of Fourier multipliers by maximal operators developed in the last chapter to obtain spectral Hörmander-Mikhlin multiplier theorems. Recall that if A is a, possibly unbounded, self adjoint operator acting on L_2, then a *spectral multiplier* is an operator of the form $m(A)$, for some function $m : \mathbb{R}_+ \to \mathbb{C}$. Bounding such operators is trivial in L_2. Our Hörmander-Miklin theorems will give (complete) L_p-boundedness for every $1 < p < \infty$, provided that m has a finite number of integrable derivatives. Such results are well known in the abelian case and generally geometrical conditions —like Sobolev-type inequalities— have to be imposed in $A : \mathrm{dom}(A) \subset L_2(X) \to L_2(X)$ to ensure the finite dimensionality of X. Here the operator A will be the generator of the Markovian semigroup of Fourier multipliers $S_t = T_{e^{-t\,\psi}}$, where $\psi : G \to \mathbb{R}_+$ is a c.n. function. Following the intuition already sketched in Section 1.10, $\mathcal{S}$ has to satisfy the ultracontractivity property $\mathrm{CBR}_{\Phi}^{2,\infty}$ and, in a way completely analogous to the classical case, we need to impose Φ to be doubling. On top of that, stronger assumptions are required here. In fact, in the classical abelian setting, apart from the ultracontractivity —or on-diagonal behaviour of S_t— we need to impose off-diagonal decay on the kernels of S_t, typically Gaussian bounds. In order to express the Gaussian bounds we will introduce an unbounded element X in the extended positive cone $\mathcal{L}G_+^\wedge$, see [Haa79a, Haa79b], describing the metric. Such X will play the role of the unbounded function $\chi \mapsto d(e, \chi)$ in the classical case. Apart from the Gaussian bounds we will ask the Hardy-Littlewood maximal associated to the metric X to be completely bounded. When such properties hold we will say that the triple $(\mathcal{L}G, \mathcal{S}, X)$ satisfies the *standard assumptions*. It is also interesting to point out that, in the classical case, Gaussian bounds can be deduced from the ultracontractivity in the presence of geometrical assumptions like locality or finite speed of propagation for the wave equation, see [Sik96, Sik04], [SC09, Section 3] and the discussion on Section 3.5. Generalizing such results to the noncommutative setting will be the object of forthcoming research.

We manage to prove that if G and ψ admit an X satisfying the standard assumptions, then, for $s > (D_\Phi + 1)/2$, we have that

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t \geq 0} \|\eta(\cdot)\, m(t\,\cdot)\|_{W^{2,s}(\mathbb{R}_+)} \quad \forall\, 1 < p < \infty,$$

see Theorem 3.2.1. Recall that the critical smoothness order is $(D_\Phi + 1)/2$ instead of the expected $D_\Phi/2$. In the Subsection 3.3 we introduce the *completely bounded q-Plancherel property* CBPlan_q^Φ in order to lower down such smoothness order to $D_\Phi/2$. Property CBPlan_q^Φ plays the role of the q-Plancherel property introduced by Duong-Ouhabaz-Sikora [DOS02]. Using that property we manage to prove that. For every $(\mathcal{L}G, \mathcal{S}, X)$ with the standard assumptions and $s > D_\Phi/2$ we have that

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t \geq 0} \|\eta(\cdot)\, m(t\,\cdot)\|_{W^{\infty,s}(\mathbb{R}_+)}, \quad \forall\, 1 < p < \infty.$$

If, in addition, ψ has the CBPlan_q^Φ, then

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t \geq 0} \|\eta(\cdot)\, m(t\, \cdot)\|_{W^{q,s}(\mathbb{R}_+)}, \quad \forall\, 1 < p < \infty.$$

See Theorem 3.3.7 for the details.

The existence of natural triples satisfying the standard assumptions for nonabelian groups is the subject of current research, which will appear elsewhere. In this Chapter we shall construct such triples out of finite-dimensional cocycles. This permits to recover the results in [JMP14a, JMP15] for ψ-radial multipliers. In fact, we should emphasize at this point that the notion of dimension in the previous approach was limited to the Hilbert space dimension of the cocycle determined by the length ψ.

The results here exposed correspond roughly with that of the second block of [GPJP15].

3.1 Standard assumptions

Let $\mathcal{L}G_+^\wedge$ denote the extended positive cone of $\mathcal{L}G$. Like in the discussion at the beginning of Section 2.2 we shall treat positive unbounded operators X affiliated to $\mathcal{L}G$ as invariant noncommutative or quantized metrics over $\mathcal{L}G$. Indeed, throughout this chapter our metric will be an element $X \in \mathcal{L}G_+^\wedge$, the extended positive cone of $\mathcal{L}G$. Note that if G is LCH and abelian, any translation-invariant metric over its dual group can be associated with the positive function $\Delta : \chi \mapsto d(\chi, e)$. The metric conditions impose that Δ is symmetric, does not vanish outside e and $\Delta(\chi_1 \chi_2) \leq \Delta(\chi_1) + \Delta(\chi_2)$. Here we will only require X to be symmetric, i.e.: to satisfy $\sigma X = X$. Recall that the anti-automorphism σ extends to $\mathcal{L}G_+^\wedge$. Following the intuition relating symmetric operators in $\mathcal{L}G_+^\wedge$ to metrics, we will say that $X \in \mathcal{L}G_+^\wedge$ is doubling iff the function $\Phi_X(r) = \tau(\chi_{[0,r)}(X))$ is doubling. When the dependency on the operator X can be understood from the context we will just write Φ. In a similar fashion, we will say that X satisfies the L_p-Hardy-Littlewood maximal property when

$$\Big\| {\sup_{r \geq 0}}^+ \Big\{ \frac{\chi_{[0,r)}(X)}{\Phi_X(r)} \star u \Big\} \Big\|_p \lesssim \|u\|_p, \tag{HL$_p$}$$

If we say that X has the HL property, omitting the dependency on p, we mean that the HL property is satisfied for every $1 < p \leq \infty$, with constants depending on p. When the property HL_p holds uniformly for all matrix amplifications, we will say that X satisfies the *completely bounded Hardy-Littlewood maximal property* (CBHL_p in short). Let $\psi : G \to \mathbb{R}_+$ be a conditionally negative length generating a semigroup $\mathcal{S}$. We will say that $\mathcal{S}$ has L_2 Gaussian bounds with respect to X when there is some $\beta > 0$ such that

$$\tau\Big\{\chi_{[r,\infty)}(X) |\lambda(e^{-t\psi})|^2\Big\} \lesssim \frac{e^{-\beta \frac{r^2}{t}}}{\Phi_X(\sqrt{t})}. \tag{L_2GB}$$

Definition 3.1.1. A triple $(\mathcal{L}G, \mathcal{S}, X)$, where $\mathcal{S}$ is a Markov semigroup of Fourier multipliers generated by $\psi : G \to \mathbb{R}_+$ and $X \in (\mathcal{L}G)_+^\wedge$, is said to satisfy the *standard assumptions* when

i) X is symmetric and doubling.

ii) $\mathcal{S}$ has L_2GB with respect to X.

iii) X satisfies the CBHL property.

Since $\mathcal{L}G$ is determined by G and $\mathcal{S}$ by ψ we shall often write (G, ψ, X) instead.

Remark 3.1.1. If $\mathcal{S}$ has L_2GB then it admits $\mathrm{CBR}_{\Phi_X}^{2,\infty}$ ultracontractivity. Namely if we take $r = 0$ in (L_2GB), it follows from Theorem 1.7.1 and Remark 1.7.2. If X is in addition doubling, $\mathcal{S}$ has the whole range of ultracontractivity properties CBR_{Φ_X}.

3.1.1 Stability under Cartesian products.

It is interesting to note that the standard assumptions are stable under certain algebraic operations, the most trivial of them is probably the Cartesian product. Stability under crossed products also holds under natural conditions, see Remark 3.1.6 below as well as Chapter 4.

The only nontrivial part of proving the stability of the standard assumptions is proving the stability of the CBHL inequalities. For that end we need to use the complete positivity to prove an stability Lemma for maximal operators.

Lemma 3.1.2. *Assume that*

$$\mathcal{R}^j = (R^j_{\omega_j})_{\omega_j \in \Omega_j} : L_p(\mathcal{M}_j) \to L_p(\mathcal{M}_j; L_\infty(\Omega_j))$$

is completely positive for $j \in \{1, 2\}$. Then $\mathcal{R}^1 \otimes \mathcal{R}^2$ is also c.p. and

$$\begin{aligned} & \Big\| \mathcal{R}^1 \otimes \mathcal{R}^2 : L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2) \to L_p\big(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2; L_\infty(\Omega_1) \otimes_{\min} L_\infty(\Omega_2)\big) \Big\|_{\mathrm{cb}} \\ \lesssim & \prod_{j \in \{1,2\}} \big\| \mathcal{R}^j : L_p(\mathcal{M}_j) \to L_p(\mathcal{M}_j; L_\infty(\Omega_j)) \big\|_{\mathrm{cb}}. \end{aligned}$$

Proof. (of Theorem 3.1.2) It follows from $\mathcal{R}^1 \otimes \mathcal{R}^2 = (\mathcal{R}^1 \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathcal{R}^2)$ and (1.8.1), details are omitted. □

Now, all we have to see is that, when $\Omega_1 = \Omega_2 = \Omega$ we can restrict to the diagonal of $\Omega \times \Omega$. Notice that the multiplication operator $m : L_\infty(\Omega) \otimes_{\min} L_\infty(\Omega) \to L_\infty(\Omega)$ given by $m(f \otimes g) = f\, g$ is completely bounded since $L_\infty(\Omega) = C(X)$, where X is the Gel'fand spectrum of $L_\infty(\Omega)$, $L_\infty(\Omega) \otimes_{\min} L_\infty(\Omega) = C(X \times X)$ and m is just the restriction to the diagonal of $K \times K$.

Lemma 3.1.3. *Let $\mathcal{A}$ be an abelian C^*-algebra and $(\mathcal{M}, \tau)$ a hyperfinite von Neumann algebra with a n.s.f. trace τ, then*

$$\big\| (\mathrm{Id}_{\mathcal{M}} \otimes m) : L_p(\mathcal{M}; \mathcal{A} \otimes_{\min} \mathcal{A}) \to L_p(\mathcal{M}; \mathcal{A}) \big\|_{\mathrm{cb}} \leq 1,$$

where $m : \mathcal{A} \otimes_{\min} \mathcal{A} \to \mathcal{A}$ is given by $f \otimes g \mapsto f\, g$.

We can pass to the proof of the main stability result.

Theorem 3.1.4. *Let (G_j, ψ_j, X_j) be triples satisfying the standard assumptions for $j = 1, 2$ and consider the Cartesian product $G = G_1 \times G_2$. Then (G, ψ, X) also satisfies the standard assumptions with the c.n. length $\psi(g_1, g_2) = \psi_1(g_1) + \psi_2(g_2)$ and $X \in \mathcal{L}G_+^\wedge$ determined by the formula $X^2 = X_1^2 \otimes \mathbf{1} + \mathbf{1} \otimes X_2^2$.*

Proof. Proving that X is doubling and that the semigroup generated by ψ has Gaussian bounds amount to a trivial calculation. Indeed, Φ_X is controlled from the inequalities $\chi_{[0,r/2)}(a)\chi_{[0,r/2)}(b) \leq \chi_{[0,r)}(a+b) \leq \chi_{[0,r)}(a)\,\chi_{[0,r)}(b)$, which are valid for positive and commuting operators a, b. On the other hand, the L_2GB follow similarly from the inequality $\chi_{[r,\infty)}(a+b) \leq \chi_{[r/2,\infty)}(a) + \chi_{[r/2,\infty)}(b)$. Let us now justify the CBHL property. Let $m : L_p(\mathcal{L}G; L_\infty \otimes_{\min} L_\infty) \to L_p(\mathcal{L}G; L_\infty)$ be the map given by $m(x \otimes f \otimes g) = x \otimes fg$, which is c.p. By Lemma 3.1.2 and Lemma 3.1.3

$$\mathcal{R}^1 \otimes \mathcal{R}^2 = \big(R^1_s \otimes R^2_t\big)_{s,t\geq 0} : L_p(\mathcal{L}G) \to L_p\big(\mathcal{L}G; L_\infty(ds) \otimes_{\min} L_\infty(dt)\big),$$

where $R^j_s(x) = \Phi_{X_j}(s)^{-1}\,\chi_{[0,s)}(X_j) \star x$ is c.p. As a consequence $m \circ (\mathcal{R}^1 \otimes \mathcal{R}^2)$ is also completely positive. Therefore, by the doubling property we obtain the following estimate

$$\Big(\frac{\chi_{[0,r)}(X)}{\Phi_X(r)} \star x\Big)_{r\geq 0} \lesssim_{(D_{\Phi_1}, D_{\Phi_2})} \Big(\frac{\chi_{[0,r)}(X_1)}{\Phi_{X_1}(r)} \otimes \frac{\chi_{[0,r)}(X_2)}{\Phi_{X_2}(r)} \star x\Big)_{r\geq 0} = m \circ (\mathcal{R}^1 \otimes \mathcal{R}^2)(x)$$

for $x \geq 0$. This is all what we need to reduce CBHL of X to that of X_1 and X_2. □

Remark 3.1.5. It is worth pointing out that the proof above of the stability of maximal bounds for tensor products uses the complete positivity of our generalized Hardy-Littlewood maximal. The property HL_p by itself is not stable under general tensor products, although it may be in particular situations.

Remark 3.1.6. Let H and G be LHC unimodular groups and $\theta : G \to \mathrm{Aut}(H)$ be a measure preserving action. Let (H, ψ_1, X_1) and (G, ψ_2, X_2) be triples satisfying the standard assumptions. It is possible to prove that, under certain invariance conditions on X_1 and ψ_1, the semidirect product $K = H \rtimes_\theta G$ satisfies the standard assumptions for some $X \in \mathcal{L}K^\wedge_*$ and certain c.n. length function $\psi : K \to \mathbb{R}_+$ built up from X_1, X_2 and ψ_1, ψ_2 respectively. Since the techniques required to prove this result are quite involved and of independent interest, we postpone its proof to Chapter 4.

3.2 Hörmander-Mikhlin criteria

In this subsection we shall give a proof of the spectral Hörmander-Mikhlin Theorem by means of a suitably chosen max-square decomposition. Our main theorem will be the following

Theorem 3.2.1. *Let (G, ψ, X) be a triple with the standard assumptions and G unimodular, $\eta \in \mathcal{H}^\infty_0$ and $m : \mathbb{R}_+ \to \mathbb{C}$ is a function, then, for every $s > (D_\Phi + 1)/2$*

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \quad \lesssim_{(p)} \quad \sup_{t\geq 0} \|\eta(\cdot)\, m(t\,\cdot)\|_{W^{2,s}(\mathbb{R}_+)} \quad \forall\, 1 < p < \infty.$$

If instead of the CBHL *we have just the* HL *inequality, the estimate above holds jut for the $\mathcal{B}(L_p(\mathcal{L}G))$-norm.*

The key to the proof of Theorem 3.2.1 is that, if $B_t = \lambda(m\,\eta(t\,\psi))$, then

$$B_t \quad = \quad \underbrace{B_t \Big(\mathbf{1} + \frac{X^2}{t}\Big)^{\frac{\gamma}{2}} \Phi(\sqrt{t})^{\frac{1}{2}}}_{\Sigma_t} \underbrace{\Phi(\sqrt{t})^{-\frac{1}{2}} \Big(\mathbf{1} + \frac{X^2}{t}\Big)^{-\frac{\gamma}{2}}}_{M_t}. \tag{3.2.1}$$

is a square-max decomposition for $\gamma > D_\Phi/2$. Breaking the symbol m into its real and imaginary parts and using Remark 2.1.3, we obtain a max-square decomposition by placing the smoothing factor $(\mathbf{1} + X^2/t)^{\gamma/2}$ on the left hand side of B_t. The proof of the maximal inequality consists in expressing the maximal operator as a linear combination of Hardy-Littlewood maximal operators associated to X and apply (1.8.2). For the square estimate we will use the smoothness condition.

Lemma 3.2.2. *Assume that $F_t \in C_0(\mathbb{R}_+)$ is a family of bounded variation functions parametrized by $t > 0$. Let ∂F_t be its Lebesgue-Stjeltjies derivative and $|\partial F_t(\lambda)|$ its absolute variation, then for every doubling operator X, we have:*

$$\Big\| \sup_{t \geqslant 0}{}^{+} \Big\{ F_t(X) \star x \Big\} \Big\|_{L_p} \quad \leq \quad \Big(\sup_{t>0} \|\Phi\|_{L_1(|dF_t|)} \Big) \Big\| \sup_{r>0}{}^{+} \Big\{ \frac{\chi_{[0,r)}(X)}{\Phi(r)} \star x \Big\} \Big\|_{L_p}$$

Proof. By integration by parts we have that

$$\begin{aligned} F_t(s) &= \int_{\mathbb{R}_+} F_t(r) d\delta_s(r) \\ &= \int_{\mathbb{R}_+} F_t(r) \partial \chi_{(s,\infty)}(r) \\ &= -\int_{\mathbb{R}_+} \chi_{(s,\infty)}(r) \partial F_t(r) \quad = \quad -\int_{\mathbb{R}_+} \frac{\chi_{[0,r)}(s)}{\Phi(r)} \Phi(r) \partial F_t(r). \end{aligned}$$

By functional calculus, the same holds for $F_t(X)$. Applying (1.8.2) ends the proof. □

According to Corollary 2.1.2, the right choice for the square-max decomposition is given by $F_t(s) = |M_t|^2(s) = \Phi(\sqrt{t})^{-1}(1 + s^2/t)^{-\gamma}$. It will suffice to pick here $\gamma > D_\Phi/2$. In order to prove the finiteness of the maximal bound in (MS_p), we just need to verify the condition of Lemma 3.2.2 for this concrete function.

Lemma 3.2.3. *For any doubling Φ, we find*

$$\int_{\mathbb{R}_+} \Phi(s) \Big| \frac{d}{ds} \Big(1 + \frac{s^2}{t} \Big)^{-\frac{D_\Phi + \varepsilon}{2}} \Big| ds \lesssim_{(D_\Phi, \varepsilon)} \Phi(\sqrt{t}).$$

Proof. Changing variables $s \mapsto \sqrt{tv}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+} \Phi(s) \Big| \frac{d}{ds} \Big(1 + \frac{s^2}{t} \Big)^{-\frac{D_\Phi + \varepsilon}{2}} \Big| d\,s \quad &\sim_{(D_\Phi)} \quad \int_{\mathbb{R}_+} \Phi(s) \Big(1 + \frac{s^2}{t} \Big)^{-\frac{D_\Phi + 2 + \varepsilon}{2}} \frac{2s}{t} \, d\,s \\ &= \quad \int_{\mathbb{R}_+} \Phi(\sqrt{t}\sqrt{v}) \big(1 + v \big)^{-\frac{D_\Phi + 2 + \varepsilon}{2}} \, d\,v \\ &= \quad \Big(\int_0^1 + \sum_{k=0}^{\infty} \int_{4^k}^{4^{k+1}} \Big) \\ &= \quad A + \sum_{k=0}^{\infty} B_k. \end{aligned}$$

The monotonicity of Φ gives $A \leq \Phi(\sqrt{t})$, while its doublingness yields

$$\begin{aligned} B_k \quad &\leq \quad \Phi(\sqrt{t}) \, 2^{D_\Phi(k+1)} \int_{4^k}^{4^{k+1}} \big(1 + v \big)^{-\frac{D_\Phi + 2 + \varepsilon}{2}} \, dv \\ &\sim_{(D_\Phi)} \quad \Phi(\sqrt{t}) \, 2^{D_\Phi(k+1)} \, 2^{-(D_\Phi + \varepsilon)k} \\ &\sim_{(D_\Phi)} \quad \Phi(\sqrt{t}) \, 2^{-\varepsilon k}. \end{aligned}$$

Since the sequence of B_ks is summable, we have proved the desired estimate. □

For the estimate of the square part, let us start by extending the Gaussian bounds to the complex half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Re(z) > 0\}$. We need the following version of the Phragmen-Lindelöff theorem.

Theorem 3.2.4. ([Dav95, Theorem 3.4.8]) *If F is analytic over $\mathbb{H}$ and satisfies*

$$\begin{aligned} |F(|z|e^{i\theta})| &\lesssim \big(|z|\,\cos\theta\big)^{-\beta}, \\ |F(|z|)| &\lesssim |z|^{-\beta}\exp\big(-\alpha|z|^{-\rho}\big), \end{aligned}$$

for some $\alpha, \beta > 0$ and $0 < \rho \leq 1$, then we find the following estimate

$$|F(|z|e^{i\theta})| \lesssim_{(\beta)} \big(|z|\,\cos\theta\big)^{-\beta}\exp\Big(-\frac{\alpha\rho}{2}|z|^{-\rho}\cos\theta\Big).$$

We may now generalize the Gaussian L_2-bounds to the complex half-plane.

Proposition 3.2.5. *Let G be a unimodular group, $\psi : G \to \mathbb{R}_+$ a c.n. length and $X \in \mathcal{L}G_+^\wedge$ a doubling operator satisfying L_2GB. If we set $h_z = \lambda(e^{-z\psi})$, the following bound holds for every $z \in \mathbb{H}$*

$$\tau\Big\{\chi_{[r,\infty)}(X)|h_z|^2\Big\} \lesssim \frac{1}{\Phi(\sqrt{\Re\{z\}})}\, e^{-\frac{\beta}{2}\frac{r^2}{|z|}\frac{\Re\{z\}}{|z|}}.$$

Proof. Let x be an element of $L_2(\mathcal{L}G)$ with $\|x\|_2 \leq 1$. Assume in addition that $x = px$ for $p = \chi_{[r,\infty)}(X)$. Then we define G_x as the following holomorphic function

$$G_x(z) = e^{-\frac{z}{t}}\,\Phi(\sqrt{t})\,\tau(h_z x)^2.$$

Then, the estimate below holds in $\mathbb{H}$

$$\begin{aligned} |G_x(z)| &= e^{-\frac{\Re\{z\}}{t}}\,\Phi(\sqrt{t})|\tau(h_z x)|^2 \\ &\leq e^{-\frac{\Re\{z\}}{t}}\,\Phi(\sqrt{t})\tau(|h_z|^2) \\ &= e^{-\frac{|z|\cos\theta}{t}}\,\Phi(\sqrt{t})\,\|h_{\Re\{z\}}\|^2_{L_2(\mathcal{L}G)} \;\lesssim\; e^{-\frac{\Re\{z\}}{t}}\,\frac{\Phi(\sqrt{t})}{\Phi(\sqrt{\Re\{z\}})}. \end{aligned}$$

Note that the second identity above follows from Plancherel theorem and the last inequality from L_2GB for $r = 0$. On the other hand, since Φ is doubling it satisfies $\Phi(s(1+r)) \lesssim \Phi(s)(1+r)^{D_\Phi}$ for every $r > 0$ and

$$\begin{aligned} |G_x(z)| &\lesssim e^{-\frac{\Re\{z\}}{t}}\,\frac{\Phi(\sqrt{t})}{\Phi(\sqrt{\Re\{z\}})} \\ &\lesssim e^{-\frac{\Re\{z\}}{t}}\Big(1+\frac{\sqrt{t}}{\sqrt{\Re\{z\}}}\Big)^{D_\Phi} \;\lesssim\; \Big(\frac{t}{|z|\,\cos\theta}\Big)^{\frac{D_\Phi}{2}}, \end{aligned}$$

by using that $e^{-s^2}(1+1/s)^a \lesssim (1/s)^a$ in the last inequality. We also have

$$\begin{aligned} |G_x(|z|)| &= e^{-\frac{|z|}{t}}\,\Phi(\sqrt{t})\,\big|\tau(h_{|z|}x)\big|^2 \\ &\leq e^{-\frac{|z|}{t}}\,\Phi(\sqrt{t})\,\tau\big\{p\,h^*_{|z|}h_{|z|}\big\} \\ &\lesssim e^{-\frac{|z|}{t}}\,\frac{\Phi(\sqrt{t})}{\Phi(\sqrt{|z|})}\,e^{-\beta\frac{r^2}{|z|}} \\ &\lesssim e^{-\frac{|z|}{t}}\Big(1+\frac{\sqrt{t}}{\sqrt{|z|}}\Big)^{D_\Phi} e^{-\beta\frac{r^2}{|z|}} \;\lesssim\; \Big(\frac{t}{|z|}\Big)^{\frac{D_\Phi}{2}} e^{-\beta\frac{r^2}{|z|}}. \end{aligned}$$

The Phargmen-Lindelöf theorem allows us to combine both estimates, giving

$$|G_x(|z|e^{i\theta})| \lesssim t^{\frac{D_\Phi}{2}} \big(|z|\cos\theta\big)^{-\frac{D_\Phi}{2}} e^{-\frac{\beta r^2}{2}\frac{\cos\theta}{|z|}}.$$

Taking the supremum over all x with $\|x\|_2 \leq 1$ and $x = p\,x$ we get

$$\sup_x |G_x(z)| = e^{-\frac{\Re\{z\}}{t}} \Phi(\sqrt{t})\tau(p\,|h_z|^2),$$

Our previous estimate then yields

$$e^{-\frac{\Re\{z\}}{t}} \Phi(\sqrt{t})\,\tau(p\,|h_z|^2) \lesssim t^{\frac{D_\Phi}{2}} \big(|z|\cos\theta\big)^{-\frac{D_\Phi}{2}} e^{-\frac{\beta r^2}{2}\frac{\cos\theta}{|z|}},$$

Choosing the parameter $t \geq 0$ to be $t = \Re\{z\}$ gives the desired estimate. □

Lemma 3.2.6. *If $X \in \mathcal{L}G_+^\wedge$ is doubling and $\psi : G \to \mathbb{R}_+$ has $(L_2\mathrm{GB})$, then*

$$\tau\Big\{\Big(\mathbf{1} + \frac{X^2}{t}\Big)^\kappa |h_{t(1-i\xi)}|^2\Big\}^{\frac{1}{2}} \lesssim_{(\kappa)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}\big(1+|\xi|\big)^\kappa \quad \textit{for all} \quad \kappa > 0.$$

Proof. Writing $z = t(1-i\xi)$ in Proposition 3.2.5 gives

$$\tau\Big\{\chi_{[r,\infty)}(X)|h_z|^2\chi_{[r,\infty)}(X)\Big\} \lesssim \frac{1}{\Phi(\sqrt{t})} e^{-\frac{\beta}{2}\frac{r^2}{t}\frac{1}{(1+|\xi|^2)}}.$$

Using the spectral measure dE_X of X and since since $(1+s^2)^\kappa \lesssim_{(\kappa)} 1 + s^{2\kappa}$

$$\begin{aligned}
\tau\Big\{\Big(\mathbf{1} + \frac{X^2}{t}\Big)^\kappa |h_{t(1-i\xi)}|^2\Big\} &\lesssim_{(\kappa)} \tau\big\{|h_{t(1-i\xi)}|^2\big\} + \tau\Big\{|h_{t(1-i\xi)}|^2 t^{-\kappa} X^{2\kappa}\Big\} \\
&\lesssim \frac{1}{\Phi\left(\sqrt{t}\right)} + \underbrace{\tau\Big\{|h_{t(1-i\xi)}|^2 \int_{\mathbb{R}_+} \Big(\frac{s^2}{t}\Big)^\kappa dE_X(s)\Big\}}_{\mathrm{A}}.
\end{aligned}$$

To estimate the term A we use integration by parts

$$\begin{aligned}
\mathrm{A} &= \int_{\mathbb{R}_+} \Big(\frac{s^2}{t}\Big)^\kappa \tau\big\{|h_{t(1-i\xi)}|^2 dE_X(s)\big\} \\
&= \int_{\mathbb{R}_+} \Big(\frac{s^2}{t}\Big)^\kappa (-\partial_s)\tau\big\{|h_{t(1-i\xi)}|^2 \chi_{[s,\infty)}(X)\big\} \\
&= \int_{\mathbb{R}_+} \frac{d}{ds}\Big(\frac{s^2}{t}\Big)^\kappa \tau\big\{|h_{t(1-i\xi)}|^2 \chi_{[s,\infty)}(X)\big\}\, ds.
\end{aligned}$$

In the second line, by $-\partial_s\tau\{|h_{t(1-i\xi)}|^2\chi_{[s,\infty)}(X)\}$, we mean the Lebesgue-Stjeltjes measure associated with the increasing function $g(s) = -\tau\{|h_{t(1-i\xi)}|^2\chi_{[s,\infty)}(X)\}$ and the third line is just an application of the integration by parts formula for Lebesgue-Stjeltjes integrals. A calculation gives that

$$\begin{aligned}
\mathrm{A} &\lesssim \int_{\mathbb{R}_+} \Big(\frac{2\kappa s^{2\kappa-1}}{t^\kappa}\Big) \frac{1}{\Phi(\sqrt{t})} e^{-\frac{\beta}{2}\frac{s^2}{t}\frac{1}{(1+|\xi|^2)}}\, ds \\
&\sim_{(\kappa)} \frac{(1+|\xi|^2)^\kappa}{\Phi(\sqrt{t})} \int_{\mathbb{R}_+} s^{2\kappa-1} e^{-\frac{\beta}{2}s^2}\, ds \\
&\sim_{(\kappa)} \frac{(1+|\xi|)^{2\kappa}}{\Phi(\sqrt{t})}.
\end{aligned}$$

and that finishes the proof. □

Proposition 3.2.7. *Let $B_t = \lambda(m(\psi)\eta_1(t\psi))$ where $\eta_1(z) = \eta(z)\, e^{-z}$ for some $\eta \in \mathcal{H}_0^\infty$. Assume also that X is a doubling operator satisfying L_2GB, then the following estimate holds for every $\delta > 0$ and $\kappa > 0$*

$$\tau\Big\{\Big(\mathbf{1} + \frac{X^2}{t}\Big)^\kappa |B_t|^2\Big\}^{\frac{1}{2}} \lesssim_{(\kappa,\delta)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}} \big\|m(t^{-1}\cdot)\,\eta(\cdot)\big\|_{W^{2,\kappa+\frac{1+\delta}{2}}(\mathbb{R}_+)}.$$

Proof. By Fourier inversion formula

$$m(s)\,\eta_1(ts) = \underbrace{m(s)\,\eta(ts)}_{m_t(ts)}\, e^{-ts} = \Big(\frac{1}{2\pi}\int_{\widehat{\mathbb{R}}} \widehat{m}_t(\xi) e^{i\xi ts}\, d\xi\Big)\, e^{-ts}.$$

Thus, by composing with ψ and applying the left regular representation

$$B_t = \frac{1}{2\pi}\int_{\widehat{\mathbb{R}}} \widehat{m}_t(\xi)\, h_{t(1-i\xi)}\, d\xi.$$

Triangular inequality for the L_2-norm with weight $(\mathbf{1} + X^2/t)$ and Lemma 3.2.6 give

$$\begin{aligned}
\tau\Big\{\Big(\mathbf{1} + \frac{X^2}{t}\Big)^\kappa |B_t|^2\Big\}^{\frac{1}{2}} \quad &= \quad \tau\Big\{\Big(\mathbf{1} + \frac{X^2}{t}\Big)^\kappa \Big|\frac{1}{2\pi}\int_{\widehat{\mathbb{R}}} \widehat{m}_t(\xi) h_{t(1-i\xi)} d\xi\Big|^2\Big\}^{\frac{1}{2}} \\
&\le \quad \frac{1}{2\pi}\int_{\widehat{\mathbb{R}}} |\widehat{m}_t(\xi)|\, \tau\Big\{\Big(\mathbf{1} + \frac{X^2}{t}\Big)^\kappa \big|h_{t(1-i\xi)}\big|^2\Big\}^{\frac{1}{2}} d\xi \\
&\lesssim_{(\kappa)} \quad \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}\int_{\widehat{\mathbb{R}}} |\widehat{m}_t(\xi)|(1+|\xi|)^{\kappa+\frac{1+\delta}{2}}(1+|\xi|)^{-\frac{1+\delta}{2}} d\xi \quad = \quad \mathrm{A}.
\end{aligned}$$

Höder's inequality in conjunction with the definition of Sobolev space then yield

$$\Phi(\sqrt{t})^{\frac{1}{2}}\mathrm{A} \quad \le \quad \Big(\int_{\widehat{\mathbb{R}}} \big(1+|\xi|\big)^{-(1+\delta)} d\xi\Big)^{\frac{1}{2}} \big\|m(t^{-1}\cdot)\,\eta(\cdot)\big\|_{W^{2,\kappa+\frac{1+\delta}{2}}(\mathbb{R}_+)}$$

The the integral above is dominated by $(1+\delta^{-1})^{\frac{1}{2}}$ and the assertion follows. □

Proof (of Theorem 3.2.1). Let $B_t = \lambda(m(\psi)\,\eta_1(t\psi))$ with $\eta_1(s) = e^{-s}\,\eta(s)$ and $B_t = \Sigma_t M_t$ be the decomposition (3.2.1) with $\gamma > D_\Phi/2$. Since we are assuming X to be symmetric, we have that $\sigma|M_t|^2 = |M_t|^2$ and, by Lemma 3.2.2 and Lemma 3.2.3, M_t satisfies the maximal inequality of (SM$_p$). By Proposition 3.2.7 we have that

$$\sup_{t>0} \|\Sigma_t\|_{L_2(\mathcal{L}G)} \quad \lesssim_{(\gamma)} \quad \sup_{t>0} \big\|m(t^{-1}\cdot)\eta(\cdot)\,\big\|_{W^{2,\gamma+\frac{1+\delta}{2}}(\mathbb{R}_+)}.$$

Therefore $B_t = \Sigma_t M_t$ is a square-max decomposition. By similar means we obtain a max-square decomposition $B_t = M_t\Sigma_t$. Since our maximal bounds trivially extend to matrix amplifications, we may apply Theorem 2.1.1 in conjunction with Remark 2.1.4 to deduce complete bounds of our multiplier $T_{m\circ\psi}$ in both row and column Hardy spaces. Finally, arguing as in Corollary 2.1.2 and noticing that $m\circ\psi \equiv m(0)$ on the subgroup $G_0 = \{g \in G : \psi(g) = 0\}$, we deduce the assertion. □

Remark 3.2.8. It is interesting to observe that the proof given here can be adapted to the classical case. Indeed, let $S_t = e^{-tA}$ be a Markovian semigroup acting on $L_\infty(X,\mu)$. Assume further that the metric measure space (X, d_Γ, μ), where d_Γ is the gradient metric [SC09, Definition 3.1], is doubling, i.e:

$$\operatorname*{ess\,sup}_{x\in X}\sup_{r>0}\Big\{\frac{\mu(B_x(2r))}{\mu(B_x(r))}\Big\} < \infty$$

and that its integral kernel $k_t(x,y)$ has Gaussian bounds with respect to the gradient distance, i.e.:

$$\big\| \chi_{[r,\infty)}(d_\Gamma(x,\cdot))\, k_t(x,\cdot) \big\|_2^2 \lesssim \frac{e^{-\beta \frac{r^2}{t}}}{\mu(B_x(\sqrt{t}))}.$$

In that case we can apply the well known covering arguments for doubling spaces to prove that the Hardy-Littlewood maximal operator is of weak type $(1,1)$ and by interpolation the HL inequalities hold. Since (X, d_Γ, μ) is a doubling metric measure space with bounded Hardy-Littlewood maximal inequalities and Gaussian Bounds we can apply the results above to reprove the classical spectral Hörmander-Mikhlin theorem as stated in [DOS02]. We shall consider this a new proof of the classical spectral Hörmander-Mikhlin. Interestingly, some of the steps of the proof are parallel to that of [DOS02] even when the main idea of our approach is to use maximal inequalities instead of Calderón-Zygmund estimates for the kernels.

3.3 The q-Plancherel condition

Notice that in the hypothesis of Theorem 3.2.1 we have a critical smoothness order which is $1/2$ larger that we will expect from the classical case of $\mathbb{R}^n$. This feature also happens in the (commutative) spectral Hörmander-Mikhlin multiplier theorems. Here we are going to show that the critical smoothness order can always be lowered to $D_\Phi/2$ paying the price of a higher integrability condition. We will also introduce an c.b. version of the spectral p-Plancherel inequality introduced in [DOS02]. Provided that such inequality holds the integrability can be lowered back to L_∞ to L_p.

Definition 3.3.1. Let $(\mathcal{M}, \tau)$ be a noncommutative measure space and let $\mathcal{S}$ be a submarkovian semigroup generated by A. We say that $\mathcal{S}$ satisfies the completely bounded q-Plancherel condition, denoted by CBPlan_q^Φ, where Φ is some increasing function and $q \in (2,\infty]$, whenever

$$\|F(A)\|_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} \lesssim \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}} \|F(t^{-1}\,\cdot)\|_{L_q(\mathbb{R}_+)}, \qquad (\mathrm{CBplan}_p^\Phi)$$

for every $t > 0$ and for every function $F : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathrm{supp}(F) \subset \big[0, t^{-1}\big]$.

Remark 3.3.1. In the context of this paper $\mathcal{M} = \mathcal{L}G$ for some LCH unimodular group G endowed with its canonical trace and $\mathcal{S} = (T_{e^{-t\psi}})_{t\geq 0}$ is a semigroup of convolution type. In that case $F(A) = T_{F(\psi)}$ and by Theorem 1.7.1 and Remark 1.7.2 we have that

$$\begin{aligned} \|T_{F(\psi)}\|_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} &= \|T_{F(\psi)}\|_{\mathcal{CB}(L_2^r(\mathcal{L}G),\mathcal{L}G)} \\ &= \|T_{F(\psi)}\|_{\mathcal{CB}(L_2^c(\mathcal{L}G),\mathcal{L}G)} = \|F(\psi)\|_{L_2(G)}. \end{aligned}$$

Thus, the CBPlan_q^Φ condition can be restated as a bound on the $\mathcal{CB}(L_2^\dagger(\mathcal{L}G),\mathcal{L}G)$ norm, where $\dagger$ is either the column or the row o.s.s. of $L_2(\mathcal{L}G)$, or as a bound in the $L_2(G)$-norm of the symbol $F(\psi)$. Furthermore, since ψ determines $\mathcal{S}$ we will sometimes say that ψ has the CBPlan_q^Φ.

For every F with $\mathrm{supp}(F) \subset \big[0, t^{-1}\big]$ we have that $F(t^{-1}\,\cdot)$ is supported in $[0,1]$. Using that $L_q([0,1]) \subset L_p([0,1])$, with contractive inclusion, we see that $\mathrm{CBPlan}_p^\Phi \Rightarrow \mathrm{CBPlan}_q^\Phi$ for $p \leq q$.

Proposition 3.3.2. *Let (G,ψ) be a pair formed by a LCH unimodular group and a c.n. length. Let Φ be a doubling function. If ψ satisfies the utracontractivity estimates $\mathrm{CBR}_\Phi^{2,\infty}$ then it satisfies* $\mathrm{CBPlan}_\infty^\Phi$.

Proof. We pick $s > 0$, to be chosen later, and notice that

$$F(\psi(g)) = F(\psi(g))\, e^{s\psi(g)}\, e^{-s\psi(g)} = G_s(\psi(g))\, e^{-s\psi}$$

where G_s is a bounded function with $\|G_s\|_\infty \leq \|F\|_\infty e^{s/t}$. Therefore

$$\begin{aligned} \|T_{F(\psi)}\|_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} &= \|T_{G_s(\psi)} S_s\|_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} \\ &\leq \|T_{G_s(\psi)}\|_{\mathcal{CB}(L_2(\mathcal{L}G))} \|S_s\|_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} \lesssim \|F\|_\infty\, e^{\frac{s}{t}}\, \Phi(\sqrt{s})^{-\frac{1}{2}}. \end{aligned}$$

Making $s = t$ and noticing that $\|F\|_\infty = \|F(t^{-1}\,\cdot)\|_\infty$ gives the desired result. □

The terminology of the q-Plancherel condition comes from the so-called spectral Plancherel measures which arise in the study of spectral properties of infinitesimal generators of Markovian semigroups over some measure spaces [Sik96, DOS02]. In the case of a semigroup of Fourier multipliers generated by a c.n. length we can define the Plancherel measure μ_ψ, as the only σ-finite measure over $\mathbb{R}_+$ satisfying that for every $F \in C_c(\mathbb{R}_+)$

$$\|T_{F(\psi)}\|_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} = \Big(\int_{\mathbb{R}_+} |F(s)|^2\, d\mu_\psi(s) \Big)^{\frac{1}{2}}. \tag{3.3.1}$$

It is trivial to see that $d\mu_\psi(r) = \partial_r \mu(\{g \in G : \psi(g) \leq r\})$, where ∂_r represents the Lebesgue-Stjeltjes derivative of the increasing function $g(r) = \mu(\{g \in G : \psi(g) \leq r\})$.

3.3.1 Characterization of the q-Plancherel condition

By formula (3.3.1) the $\mathcal{CB}(L_2(\mathcal{L}G), \mathcal{L}G)$ norm of $T_{F(\psi)}$ can be expressed as an integral of F. The following lemma (whose proof is straightforward and we shall omit) allows to express the CBPlan_q^Φ property as a $L_{(q/2)'}(\mathbb{R}_+)$ bound on μ_ψ.

Lemma 3.3.3. *Let (Ω, Σ) be a measurable space and consider two measures μ, ν on it. Assume in addition that μ is a positive measure. Then, we have the inequality*

$$\Big| \int_\Omega f(\omega) d\nu(\omega) \Big| \leq K \|f\|_{L_p(d\mu)} \tag{3.3.2}$$

if and only if $\nu \ll \mu$ and $\phi = d\nu/d\mu$ satisfies $\|\phi\|_{L_{p'}(d\mu)} \leq K$. Furthermore, the optimal K in (3.3.2) *is precisely $\|\phi\|_{L_{p'}(d\mu)}$. If ν is also positive, it is enough for* (3.3.2) *to hold only for positive functions.*

Proposition 3.3.4. *Let G be a LCH unimodular group equipped with a c.n. length $\psi : G \to \mathbb{R}_+$. Then, this pair satisfies the* CBPlan_q^Φ *property with respect to some increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ if and only if $d\mu_\psi(r) = \partial_r \mu\{g \in G : \psi(g) \leq r\}$ fulfills the following conditions*

i) $d\mu_\psi \ll dm$.

ii) $\Big\| \dfrac{d\mu_\psi}{dm} \chi_{[0,R]} \Big\|_{L_{(q/2)'}(\mathbb{R}_+)} \lesssim \Phi(R^{-\frac{1}{2}})^{-1} R^{-\frac{2}{q}}$ *for every $R > 0$.*

Proof. Let $t = 1/R$ and $G(s) = |F(s)|^2$. By (3.3.1), CBPlan_q^Φ is equivalent to

$$\int_0^R G(s)\, d\mu_\psi(s) \quad \lesssim \quad \Phi(R^{-\frac{1}{2}})^{-1} \Big(\int_0^1 |F(t^{-1}s)|^q\, ds \Big)^{\frac{2}{q}}$$

$$= \quad \Phi(R^{-\frac{1}{2}})^{-1} R^{-\frac{2}{q}} \Big(\int_0^R |G(s)|^{\frac{q}{2}} \, d\, s \Big)^{\frac{2}{q}}.$$

Then, the result follows applying Lemma 3.3.3 to $(\Omega, d\nu, d\mu) = (\mathbb{R}_+, d\mu_\psi, dm)$. □

The result above uses the crucial fact that the spectrum of the semigroup $\mathcal{S}$ generated by ψ can be identified with G. Therefore, spectral properties of the semigroup can be translated into geometrical properties of G. It is also interesting to note that the characterization in Proposition 3.3.4 can be expressed as a bound for the size of the spheres associated to the pseudo-metric $d_\psi(g,h) = \psi(g^{-1}h)^{1/2}$.

3.3.2 Stability under direct products

Consider two pairs (G_j, ψ_j) of LCH unimodular groups equipped with c.n. lengths for $j = 1, 2$. Then it is clear that $\psi : G_1 \times G_2 \to \mathbb{R}_+$ given by $\psi(g,h) = \psi_1(g) + \psi_2(h)$ is also a c.n. length. Notice that

$$\begin{aligned} \big\| T_{F(\psi)} \big\|^2_{\mathcal{CB}(L_2(\mathcal{L}G), \mathcal{L}G)} &= \int_{G_1 \times G_2} \big| F(\psi_1(g) + \psi_2(h)) \big|^2 \, d\mu_{G_1}(g) \, d\mu_{G_2}(h) \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \big| F(\xi + \zeta) \big|^2 \, d\mu_{\psi_1}(\xi) \, d\mu_{\psi_2}(\zeta) \\ &= \int_{\mathbb{R}_+} |F(\xi)|^2 d\, (\mu_{\psi_1} * \mu_{\psi_2})(\xi). \end{aligned}$$

Thus, the Plancherel measure is $\mu_\psi = \mu_{\psi_1} * \mu_{\psi_2}$ and we obtain the following result.

Theorem 3.3.5. *Assume (G_j, ψ_j) satisfy $\mathrm{CBPlan}_{q_j}^{\Phi_j}$ for $j = 1, 2$. Then the pair $(G_1 \times G_2, \psi)$ defined above satisfies the CBPlan_q^{Φ} property with $\Phi = \Phi_1 \Phi_2$ and with*

$$q = \max \Big\{ 2, \Big(\frac{1}{q_1} + \frac{1}{q_2} \Big)^{-1} \Big\}.$$

Proof. The result is a simple consequence of Young's inequality for convolutions and we shall just sketch the argument for the (slightly more involved) case where $1/q_1 + 1/q_2 > 1/2$, so that $q = 2$. According to Proposition 3.3.4, it suffices to see that

$$\Big\| \frac{d\psi_1}{dm} * \frac{d\psi_2}{dm} \Big\|_{L_\infty(0,R)} \leq \frac{1}{R \, \Phi_1(R^{-1/2}) \Phi_2(R^{-1/2})}.$$

The $\mathrm{CBPlan}_{q_1}^{\Phi_1}$ property of (G_1, ϕ_1) implies

$$\Big\| \frac{d\psi_1}{dm} * \frac{d\psi_2}{dm} \Big\|_\infty \leq \Big\| \frac{d\psi_1}{dm} \Big\|_{(\frac{q_1}{2})'} \Big\| \frac{d\psi_2}{dm} \Big\|_{\frac{q_1}{2}} \leq \frac{1}{R^{\frac{2}{q_1}} \, \Phi_1(R^{-1/2})} \Big\| \frac{d\psi_2}{dm} \Big\|_{\frac{q_1}{2}}.$$

Now, since $1/q_1 + 1/q_2 > 1/2$ it turns out that

$$\frac{1}{q_1/2} = \frac{1}{(q_2/2)'} + \frac{1}{r} \Rightarrow \Big\| \frac{d\psi_2}{dm} \Big\|_{\frac{q_1}{2}} \leq R^{\frac{1}{r}} \Big\| \frac{d\psi_2}{dm} \Big\|_{(\frac{q_2}{2})'}.$$

The result follows from the characterization of $\mathrm{CBPlan}_{q_1}^{\Phi_1}$ in Proposition 3.3.4. □

Remark 3.3.6. A result along the same lines can be obtained for crossed products under invariance assumptions on ψ_1. This goes in the same spirit as Remark 3.1.6.

3.3.3 Refinement of the smoothness condition

Here we are going to see how we can improve the Hörmander-Mikhlin condition of Theorem 3.2.1 when ψ satisfies the CBPlan_q^Φ property.

Theorem 3.3.7. *Let (G, ψ, X) be a triple with the standard assumptions, $\eta \in \mathcal{H}_0^\infty$ a cut-off function and $m : \mathbb{R}_+ \to \mathbb{C}$, then for every $s > D_\Phi/2$*

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t\geq 0} \|\eta(\cdot)\, m(t\,\cdot)\|_{W^{\infty,s}(\mathbb{R}_+)} \quad \forall\, 1 < p < \infty. \tag{3.3.3}$$

Furthermore if ψ has the CBPlan_q^Φ

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t\geq 0} \|\eta(\cdot)\, m(t\,\cdot)\|_{W^{q,s}(\mathbb{R}_+)} \quad \forall\, 1 < p < \infty. \tag{3.3.4}$$

We need several preparatory lemmas. In the next one we denote by $W_\eta^{p,s}(\mathbb{R}_+)$, where $\eta \in \mathcal{H}_0^\infty$, the Sobolev space given by completion with respect to the norm

$$\|f\|_{W_\eta^{p,s}(\mathbb{R}_+)} = \big\|(1-\partial_x^2)^{s/2}(\eta f)\big\|_p.$$

Lemma 3.3.8. *Given $f, g : \mathbb{R}_+ \to \mathbb{C}$, the following holds:*

i) *For every $\varepsilon > 0$*

$$\big\|(1-\partial_x^2)^{s/2}(f\,g)\big\|_2 \lesssim_{(s,\varepsilon)} \big\|(1-\partial_x^2)^{(s+1+\varepsilon)/2} f\big\|_\infty \big\|(1-\partial_x^2)^{s/2} g\big\|_2.$$

ii) *If $\rho(z) = z^s e^{-z}$ and $\eta \in \mathcal{H}_0^\infty$*

$$\big\|(1-\partial_x^2)^{s/2}(\eta\,\rho\, f)\big\|_2 \lesssim_{(s,\varepsilon)} \big\|(1-\partial_x^2)^{(s+1+\varepsilon)/2}(\eta\, f)\big\|_\infty.$$

Equivalently, we find the embedding $W_\eta^{\infty,s+1+\varepsilon}(\mathbb{R}_+) \subset_{(s,\varepsilon)} W_{\eta\rho}^{2,s}(\mathbb{R}_+)$.

The proof requires the following Lemma, which follows after applying the Hilbert transform or identity map to [Ste70, V.(26)].

Lemma 3.3.9. *Let $\nu_\alpha \in M(\mathbb{R})$ be a finite measure such that*

$$\widehat{\nu}_\alpha(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{\alpha}{2}}}.$$

If $\alpha > 1$ then $H(\nu_\alpha)$ is also a finite measure, where H is the Hilbert transform.

Proof. The second point follows immediately from the first one by noticing that $\rho(z) = z^s e^{-z}$ has finite $W^{2,s}(\mathbb{R}_+)$ norm. We are going to prove the first point for $s \in \mathbb{N}$ and use interpolation. Given $s \in \mathbb{N}$, we have

$$\begin{aligned}
\big\|(1-\partial_x^2)^{s/2}(fg)\big\|_2 &\sim \sum_{k=0}^{s} \|\partial_x^k(fg)\|_2 \\
&= \sum_{k=0}^{s} \Big\| \sum_{j=0}^{k} \binom{k}{j} (\partial_x^j f)(\partial_x^{k-j} g) \Big\|_2 \\
&\lesssim_{(s)} \Big(\max_{0\leq j\leq s} \|\partial_x^j f\|_\infty\Big)\Big(\sum_{k=0}^{s} \|\partial_x^k g\|_2\Big) \\
&\sim \Big(\max_{0\leq j\leq s} \|\partial_x^j f\|_\infty\Big) \big\|(1-\partial_x^2)^{s/2} g\big\|_2.
\end{aligned}$$

Thus, all we have to see is that for every $j \in \{0, 1, 2, ..., s\}$

$$\big\|\partial_x^j (1-\partial_x^2)^{-(s+\varepsilon+1)/2} f\big\|_\infty \lesssim_{(s,\varepsilon)} \|f\|_\infty.$$

Recall that if the symbol of a Fourier multiplier is given by the Fourier transform of finite measure, then it is bounded in $L_\infty(\mathbb{R})$. Thus, we just need to see that there is a finite measure $\mu_{j,s}$ such that

$$\begin{aligned}\widehat{\mu}_{j,s}(\xi) &= \frac{\xi^j}{(1+|\xi|^2)^{\frac{s+\varepsilon+1}{2}}}\\ &= \mathrm{sgn}(\xi)^j \, \frac{1}{(1+|\xi|^2)^{\frac{s+\varepsilon-j+1}{2}}} \, \frac{|\xi|^j}{(1+|\xi|^2)^{\frac{j}{2}}} \quad = \quad (H_{[j]}(\nu_{s,j}) * m_j)^\wedge(\xi),\end{aligned}$$

where $H_{[j]}$ is the Hilbert transform for j odd and the identity map for j even. By [Ste70, V.3, Lemma 2] m_j is a finite measure. Therefore, by Lemma 3.3.9, since $\widehat{\nu}_{s,j}(\xi) = 1/(1+|\xi|^2)^{(s+\varepsilon-j+1)/2}$, $H_{[j]}(\nu_{s,j})$ is a finite measure. □

Lemma 3.3.10. *Assume G is a LCH unimodular group, $\psi : G \to \mathbb{R}_+$ is a c.n. length and that they satisfy the CBPlan_q^Φ property. If η_1, $\eta_2 \in \mathcal{H}_0^\infty(\Sigma_\theta)$, with η_1 satisfying that there is $\gamma > 0$ such that $|\eta_1(z)| \lesssim e^{-\gamma \Re(z)}$ for all $z \in \Sigma_\theta$, then the following estimate holds for all $m \in L_\infty(\mathbb{R}_+)$*

$$\big\|\lambda\big(m(\psi)\,\eta_1(t\psi)\,\eta_2(t\psi)\,\big)\big\|_{L_2(\mathcal{L}G)} \lesssim_{(D_\Phi,q,\gamma)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}} \,\big\|m(t^{-1}\,\cdot)\eta_2(\cdot)\big\|_{L_q(\mathbb{R}_+)}.$$

Proof. Using integration by parts we obtain

$$\begin{aligned}\big\|\lambda\big(m(\psi)\eta_1(t\psi)\eta_2(t\psi)\big)\big\|_{L_2(\mathcal{L}G)} &= \Big\|\int_{\mathbb{R}_+} \lambda\big(m(\psi)\eta_1(r)\,\eta_2(t\psi)\,\chi_{[0,r)}(t\psi)\big)\,dr\Big\|_{L_2(\mathcal{L}G)}\\ &\leq \int_{\mathbb{R}_+} \eta_1'(r)\big\|\lambda\big(m(\psi)\,\eta_2(t\psi)\,\chi_{[0,r)}(t\psi)\big)\big\|_{L_2(\mathcal{L}G)}\,dr.\end{aligned}$$

Now, applying the CBPlan_q^Φ property, we obtain

$$\begin{aligned}\big\|\lambda\big(m(\psi)\,\eta_1(t\psi)\,\eta_2(t\psi)\big)\big\|_{L_2(\mathcal{L}G)} &\lesssim_{(q)} \int_{\mathbb{R}_+} \eta_1'(r)\frac{1}{\Phi(\sqrt{t/r})^{\frac{1}{2}}}\big\|m((r/t)\cdot)\,\eta_2(r\cdot)\big\|_{L_q([0,1])}\,dr\\ &= \Big(\int_{\mathbb{R}_+} \eta_1'(r)\,\frac{r^{-1/q}}{\Phi(\sqrt{t/r})^{\frac{1}{2}}}\,dr\Big)\,\big\|m(t^{-1}\,\cdot)\,\eta_2(\cdot)\big\|_{L_q(\mathbb{R}_+)}.\end{aligned}$$

So, we just need to estimate the integral in the right hand side term

$$\begin{aligned}\int_{\mathbb{R}_+} \eta_1'(r)\frac{r^{-1/q}}{\Phi(\sqrt{t/r})^{\frac{1}{2}}}\,dr &= \Big\{\int_0^1 + \sum_{j=0}^\infty \int_{4^j}^{4^{j+1}}\Big\}\,\eta_1'(r)\,\frac{r^{-1/q}}{\Phi(\sqrt{t/r})^{\frac{1}{2}}}\,dr\\ &= A + \sum_{j=0}^\infty B_j.\end{aligned}$$

The first term is bounded as follows

$$\begin{aligned}A &\leq \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}\int_0^1 \eta_1'(r)\,r^{-1/q}dr\\ &\lesssim_{(q)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}.\end{aligned}$$

For the rest of the terms, we apply the doubling condition to obtain

$$\begin{aligned} B_j &= \int_{4^j}^{4^{j+1}} \eta_1'(r) \frac{r^{-1/p}}{\Phi(\sqrt{t/r})^{\frac{1}{2}}} dr \\ &\leq (4^{j+1} - 4^j) \, \|\eta_1'\|_{L_\infty([4^j, 4^{j+1}))} \frac{2^{\frac{D_\Phi}{2}(j+1)}}{\Phi(\sqrt{t})} \\ &= \frac{3\, 2^{\frac{D_\Phi}{2}}}{\Phi(\sqrt{t})^{\frac{1}{2}}} \, \|\eta_1'\|_{L_\infty([4^j, 4^{j+1}))} \, 2^{\left(\frac{D_\Phi}{2}+2\right) j}. \end{aligned}$$

The function η_1 decreases exponentially and so does η_1'. Therefore $\eta_1'(z) \lesssim e^{-\gamma z}$ for $\Re\{z\}$ large enough. That allows us to sum up all the terms in the series obtaining $\sum_j B_j \lesssim \Phi(\sqrt{t})^{-\frac{1}{2}}$ up to a constant depending on (D_Φ, γ), as desired. □

Proposition 3.3.11. *Assume G is a LCH unimodular group, $\psi : G \to \mathbb{R}_+$ is a c.n. length and that they satisfy the* CBPlan_q^Φ *property. Assume in addition that $X \in \mathcal{L}G_+^\wedge$ is doubling and admits* L_2GB. *Then, we find for $\kappa, \delta, \varepsilon > 0$*

$$\tau\left\{ \left(\mathbf{1} + \frac{X^2}{t} \right)^\kappa |B_t|^2 \right\}^{\frac{1}{2}} \lesssim_{(D_\Phi, q, \kappa, \delta, \varepsilon)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}} \left\| m(t^{-1} \cdot) \, \eta(\cdot) \right\|_{W^{p, \kappa+\delta}(\mathbb{R}_+)},$$

where $B_t = \lambda\left(m(\psi)\eta(t\psi)e^{-2t\psi}(t\psi)^a\right)$, η is a $\mathcal{H}_0^\infty$-cut-off and $a = 2\kappa/\delta + (1+\varepsilon)/2$.

Proof. Fix $\kappa, \delta, \varepsilon > 0$ and $a = 2\kappa/\delta + (1+\varepsilon)/2$. We define the linear, unbounded map $K_t : D \subset L_\infty(\mathbb{R}_+) \to L_2(\mathcal{L}G)$ by $K_t(m) = \lambda(m(t\psi)\eta(t\psi)e^{-2t\psi}(t\psi)^a)$. Using Lemma 3.3.10 with $\eta_1(z) = z^a e^{-2z}$ and $\eta_2(z) = \eta(z)$ gives that

$$\left\| K_t : W_\eta^{q,0}(\mathbb{R}_+) \to L_2(\mathcal{L}G) \right\| \lesssim_{(D_\Phi, q)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}. \tag{3.3.5}$$

Let us denote by $\phi_{t,\kappa}$ the family of weights given by

$$\phi_{t,\kappa}(x) = \tau\left\{ \left(\mathbf{1} + \frac{X^2}{t} \right)^\kappa x \right\}$$

and let $L_2(\mathcal{L}G, \phi_{t,\kappa})$ be the Hilbert spaces associated to the GNS construction of $\phi_{t,\kappa}$. We know from Proposition 3.2.7 that

$$\left\| K_t : W_{\eta\rho}^{2, s+\frac{1+\varepsilon}{2}}(\mathbb{R}_+) \to L_2(\mathcal{L}G, \phi_{t,s}) \right\| \lesssim_{(\kappa,\delta,\varepsilon)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}},$$

where $s = 2\kappa/\delta$ and $\rho(z) = z^a e^{-z}$. Composing with the inclusion

$$W_\eta^{q, s+\frac{1+\varepsilon}{2}+1+\varepsilon'}(\mathbb{R}_+) \subset_{(s,\varepsilon')} W_{\eta\,\rho}^{2, s+\frac{1+\varepsilon}{2}}(\mathbb{R}_+),$$

which follows by interpolation from Lemma 3.3.8 for $q = \infty$ and the trivial inclusion for $q = 2$, gives

$$\left\| K_t : W_\eta^{q, s+\frac{1+\varepsilon}{2}+1+\varepsilon'}(\mathbb{R}_+) \to L_2(\mathcal{L}G, \phi_{t,s}) \right\| \lesssim_{(\kappa,\delta,\varepsilon,\varepsilon')} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}. \tag{3.3.6}$$

Notice that the spaces obtained through GNS construction $L_2(\mathcal{L}G, \phi_{t,\kappa})$ are well behaved with respect to the complex interpolation method. In particular, the expected identity below holds

$$\left[L_2\left(\mathcal{L}G, \phi_{t,\kappa_1}\right), L_2\left(\mathcal{L}G, \phi_{t,\kappa_2}\right) \right]_\theta = L_2\big(\mathcal{L}G, \phi_{t,(1-\theta)\kappa_1+\theta\kappa_2}\big).$$

Therefore, interpolating (3.3.5) and (3.3.6) with $\theta = \delta/2$ yields

$$\big\| K_t : W_\eta^{q,\kappa+\frac{\delta}{2}(\frac{1+\varepsilon}{2}+1+\varepsilon')}(\mathbb{R}_+) \to L_2(\mathcal{L}G, \phi_{t,\theta s}) \big\| \lesssim_{(D_\Phi,q,\kappa,\delta,\varepsilon,\varepsilon')} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}.$$

Finally, choosing ε and ε' such that $((1+\varepsilon)/2 + 1 + \varepsilon') \le 2$ gives

$$\big\| K_t : W_\eta^{q,\kappa+\delta}(\mathbb{R}_+) \to L_2(\mathcal{L}G, \phi_{t,\kappa}) \big\| \lesssim_{(D_\Phi,q,\kappa,\delta)} \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}}.$$

Applying this bound to the function $m(t^{-1}\cdot)$ proves the assertion. $\square$

Proof (of Theorem 3.3.7. Let $s > D_\Phi/2$. For any $\eta \in \mathcal{H}_0^\infty$ and $\delta, \varepsilon > 0$ we can define $\eta_1(z) = \eta(z)e^{-2z}z^a$, where $a = 2s/\delta + (1+\varepsilon)/2$. Set $B_t = \lambda(m(\psi)\,\eta_1(t\psi))$ and apply (3.2.1). By Proposition 3.3.11 if we have the CBPlan_q^Φ property, then

$$\sup_{t>0} \|\Sigma_t\|_{L_2(\mathcal{L}G)} \quad \lesssim_{(D_\Phi,q,s,\delta,\varepsilon)} \quad \sup_{t>0} \big\| m(t^{-1}\,\cdot)\eta(\cdot) \big\|_{W^{q,s+\delta}(\mathbb{R}_+)}.$$

Once this is settled, the argument continues as in the proof of Theorem 3.2.1 giving (3.3.4). For (3.3.3) we just use that the standard assumptions imply $\mathrm{CBPlan}_\infty^\Phi$ $\square$

3.4 An application for finite-dimensional cocycles

Our aim is to recover the main result in [JMP14a] for the case of radial multipliers to illustrate how the Sobolev dimension approach is, a priori, more flexible than the one used in [JMP14a]. We will start proving that c.n. lengths coming from surjective and proper finite-dimensional cocycles satisfy the standard assumptions. Then we will reduce the case of general finite-dimensional cocycles to surjective and proper ones.

Let $b : G \to \mathbb{R}^n$ be a finite-dimensional cocycle. Assume that b is surjective and proper (i.e. $b^{-1}[K]$ is a compact set for every compact K). Then the pullback of the Haar measure $b^*\mu(E) = \mu(b^{-1}[E])$ in $\mathbb{R}^n$ is translation invariant and therefore satisfies satisfies that $d\,b^*\mu(\xi) = c\,d\,\xi$. Indeed, let $\alpha : G \to O(\mathbb{R}^n)$ be the orthogonal action naturally associated to b. Given a Borel compact set $E \subset \mathbb{R}^n$ with $b^{-1}(E) = A \subset G$ and since $b(gA) = \alpha_g(b(A)) + b(g)$, we conclude that

$$b^*\mu(E) = \mu(A) = \mu(gA) = b^*\mu(\alpha_g(E) + b(g)).$$

Note that $\mu(A)$ is well-defined and finite since b is continuous and proper. Applying this identity to the α-invariant sets $E = B_r(0)$ and using the subjectivity of b, we conclude the assertion.

An important consequence of this fact is that

$$\|S_t\|^2_{\mathcal{CB}(L_2(\mathcal{L}G),\mathcal{L}G)} = \int_G |e^{-t\psi(g)}|^2\, d\mu(g) = \int_{\mathbb{R}^n} e^{-2t|\xi|^2} d\,(b^*\mu)(\xi) = \frac{1}{\Phi(\sqrt{t})},$$

where $\mathcal{S} = (S_t)_{t\ge 0}$ is the semigroup associated with $\psi(g) = \|b(g)\|^2$ and $\Phi(t) \sim t^n$. Therefore, the semigroup associated to any proper and surjective finite-dimensional cocycle satisfies the CBR^Φ property. In the same way, the measure μ_ψ defined in (3.3.1) can be expressed (using polar coordinates) as in terms of $b^*\mu$ and a trivial calculation gives that ψ has the CBPlan_2^Φ property.

We need to find a suitable $X_b \in \mathcal{L}G_+^\wedge$. We shall prove that b induces a natural transference map from functions $f : \mathbb{R}^n \to \mathbb{C}$ into operators $x \in \mathcal{L}G$ given by

$$\mathcal{J}(f) = \lambda(\widehat{f} \circ b).$$

Therefore, if $\mathcal{R}$ is a distribution in $\mathbb{R}^n$ such that $\widehat{\mathcal{R}}(x) = |x|$, our choice will be $X_b = \lambda(\mathcal{R}(b))$.

Before proving $X_b \in \mathcal{L}G_+^\wedge$ we will need the following auxiliary result.

Lemma 3.4.1. *If $\varphi_j : \mathbb{R}^n \to \mathbb{C}$ are radial L_1-functions*

$$\lambda(\varphi_1 \circ b)\, \lambda(\varphi_2 \circ b) = \lambda\big((\varphi_1 * \varphi_2) \circ b\big)$$

for any group G equipped with a proper and surjective cocycle $b : G \to \mathbb{R}^n$.

Proof. Up to constants, we know that $d(b^*\mu) = dm$, so that

$$\begin{aligned}
(\varphi_1 \circ b) * (\varphi_2 \circ b)(g) &= \int_G \varphi_1(b(h))\varphi_2(b(h^{-1}g))\, d\mu(h) \\
&= \int_G \varphi_1(b(h))\varphi_2(\alpha_{h^{-1}}(b(g) - b(h)))\, d\mu(h) \\
&= \int_G \varphi_1(b(h))\varphi_2(b(g) - b(h))\, d\,\mu(h) \\
&= \int_{\mathbb{R}^n} \varphi_1(\zeta)\varphi_2(b(g) - \zeta) d\,(b^*\mu)(\zeta) \\
&= \int_{\mathbb{R}^n} \varphi_1(\zeta)\varphi_2(b(g) - \zeta) d\,\zeta \\
&= (\varphi_1 * \varphi_2)(b(g)).
\end{aligned}$$

Taking the left regular representation at both sides yields the assertion. □

It is straightforward to restate Lemma 3.4.1 in terms of the transference operator $\mathcal{J}$. We shall be working with the following subclass of radial functions in the Euclidean space $\mathbb{R}^n$

$$A(\mathbb{R}^n)_{\mathrm{rad}} = \big\{ \phi : \mathbb{R}^n \to \mathbb{C} \,\big|\, \phi \text{ radial}, \hat{\phi} \in L_1(\mathbb{R}^n) \big\}.$$

The norm on $A(\mathbb{R}^n)_{\mathrm{rad}}$ given by $\|\phi\|_{A(\mathbb{R}^n)_{\mathrm{rad}}} = \|\hat{\phi}\|_1$ makes such space a Banach algebra. If needed, $A(\mathbb{R}^n)_{\mathrm{rad}}$ will be given the o.s.s. inherited from the Fourier class $A(\mathbb{R}^n)$. Observe that Lemma 3.4.1 implies that

$$\mathcal{J}(\phi_1\, \phi_2) = \mathcal{J}(\phi_1)\, \mathcal{J}(\phi_2), \tag{3.4.1}$$

for any $\phi_j \in A(\mathbb{R}^n)_{\mathrm{rad}}$. In fact, we have the following Lemma.

Lemma 3.4.2. *Let $b : G \to \mathbb{R}^n$ be a proper and surjective cocycle. Then,*

(i) *$\mathcal{J} : A(\mathbb{R}^n)_{\mathrm{rad}} \to \mathcal{L}G$ is contractive, a $*$-homomorphism for the natural conjugation and positivity preserving.*

(ii) *For every radial function $\phi \in A(\mathbb{R}^n)_{\mathrm{rad}}$, we have that $\|\mathcal{J}(\phi)\|_{\mathcal{L}G} = \|\phi\|_\infty$.*

Proof. Let us start with (i). It is trivial that $\mathcal{J}$ is bounded since

$$\|\mathcal{J}(\phi)\|_{\mathcal{L}G} = \|\lambda(\hat{\phi} \circ b)\|_{\mathcal{L}G} \le \|\hat{\phi} \circ b\|_{L_1(G)} = \|\hat{\phi}\|_{L_1(\mathbb{R}^n)}.$$

The multiplicativity follows from (3.4.1) and the fact that the map is $*$-preserving is trivial. To see that the map is positivity preserving just notice that if $0 \leq \phi \in A(\mathbb{R}^n)_{\mathrm{rad}}$, then $\hat{\phi}$ is of positive type. But, if that is the case, then

$$\hat{\phi}(b(g^{-1} h)) = \hat{\phi}(\alpha_{g^{-1}}(b(g) - b(h))) = \hat{\phi}(b(g) - b(h))$$

and so $\hat{\phi} \circ b$ is also of positive type over G.

For (ii), let $\mathcal{M} \subset \mathcal{L}G$ be the weak-$*$ closure of $\mathcal{A} = \mathcal{J}[A(\mathbb{R}^n)_{\mathrm{rad}}]$. Clearly, since

$$\tau(\mathcal{J}(\phi)) = (\widehat{\phi} \circ b)(e) = \int_{\mathcal{R}^n} \phi(\xi) \, dm,$$

we have that $\tau|_{\mathcal{M}}$ is also semifinite and faithful. As a consequence we get

$$\|x\|_{\mathcal{M}} = \sup_{\xi \in \mathrm{Ball}(L_2(\mathcal{M}, \tau|_{\mathcal{M}}))} \|x \, \xi\|_2 \tag{3.4.2}$$

and by the Plancherel identity $L_2(\mathcal{M}) \cong L_2(\mathbb{R}^n)_{\mathrm{rad}}$. We are also going to use that if $\mathcal{A} \subset \mathcal{M}$ is a weak-$*$ dense $*$-subalgebra of $\mathcal{M}$, then $L_2(\mathcal{M}) \cap \mathcal{A}$ is norm dense inside $L_2(\mathcal{M})$. Those two facts yield that

$$\begin{aligned} \|\mathcal{J}(\phi)\|_{\mathcal{L}G} &= \sup_{\xi \in \mathrm{Ball}(L_2(\mathcal{M}))} \|\mathcal{J}(\phi) \, \xi\|_2 \\ &= \sup_{\mathcal{J}(\psi) \in \mathcal{A} \cap \mathrm{Ball}(L_2(\mathcal{M}))} \|\mathcal{J}(\phi \, \psi)\|_2 \\ &= \sup_{\psi \in \mathrm{Ball}(L_2(\mathbb{R}^n)_{\mathrm{rad}})} \|\phi \, \psi\|_2 \quad = \quad \|\phi\|_\infty, \end{aligned}$$

for any radial ϕ and that concludes the proof. □

In order to define X_b as an element of $\mathcal{L}G_+^\wedge$, we need to express it as the supremum of positive operators in $\mathcal{L}G$. We use

$$1 = \int_{\mathbb{R}_+} s \, |\xi|^2 \, e^{-s|\xi|^2} \frac{d \, s}{s},$$

and think of $\eta_s(\xi) = |\xi|^2 \, s \, e^{-s|\xi|^2}$ as a continuous partition of the unit. Hence

$$|\xi| = \int_{\mathbb{R}_+} |\xi| \, \eta_s(\xi) \frac{d \, s}{s},$$

and so, we define

$$\phi_{\varepsilon,R}(\xi) \quad := \quad \int_\varepsilon^R |\xi| \, \eta_s(\xi) \frac{d \, s}{s} \qquad \text{and} \qquad X_b \quad := \quad \sup_{0 < \varepsilon \leq R < \infty} \mathcal{J}(\phi_{\varepsilon,R}).$$

This presents X_b as a well-defined element of the extended positive cone $\mathcal{L}G_+^\wedge$.

Theorem 3.4.3. *Let G be a LCH unimodular group and consider an n-dimensional proper and surjective cocycle $b : G \to \mathbb{R}^n$ equipped with the conditionally negative length $\psi(g) = \|b(g)\|^2$. Then (G, ψ, X_b) satisfies the standard assumptions.*

Proof. We will start by proving the L_2GB. By noticing that $\zeta \mapsto \chi_{[r,\infty)}(\zeta)$ is an increasing function and the normality of the weight $x \mapsto \tau \left\{ x \, |\lambda(e^{-t\psi})|^2 \right\}$ we obtain that

$$\tau\Big\{\chi_{[r,\infty)}(X_b) \, |\lambda(e^{-t\psi})|^2\Big\} = \sup_{0 < \varepsilon \leq R < \infty} \tau\Big\{\chi_{[r,\infty)}(\mathcal{J}(\phi_{\varepsilon,R})) \, |\lambda(e^{-t\psi})|^2\Big\}.$$

If P is a polynomial, (3.4.1) gives $P(\mathcal{J}(\phi)) = \mathcal{J}(P(\phi))$. The function $\chi_{[r,\infty)}$ may not be a polynomial but we can approximate it by analytic functions as follows. Let F be

$$F(\zeta) = \frac{1}{2} + \frac{1}{\pi}\int_0^\zeta e^{-s^2}\,ds. \tag{3.4.3}$$

We define the function $\chi_{n,r} \geq 0$ by

$$\chi_{n,r}(\zeta) = \big(F(n\,(\zeta - r)) - F(-n\,r)\big)^2.$$

For $r > 0$, the positive functions $\chi_{r,n}$ converge pointwise and boundedly to $\chi_{[r,\infty)}$ as $n \to \infty$. Furthermore, $\chi_{n,r}(0) = 0$ and $\chi_{n,r}$ is a real analytic function with arbitrarily large convergence radius. By the analyticity it holds that for any radial ϕ in the Schwartz class

$$\chi_{n,r}(\mathcal{J}(\phi)) = \mathcal{J}(\chi_{r,n}(\phi)).$$

The right hand side is well-defined since $\chi_{r,n}(\phi)$ is again a Schwartz class function and so its Fourier transform is integrable. By [Fol95, Proposition 1.48] if $\chi_{n,r}$ converges to $\chi_{[r,\infty)}$ pointwise and boundedly then $\chi_{n,r}(x)$ converges to $\chi_{[0,\infty)}(x)$ is the SOT topology for any positive $x \in \mathcal{L}G$. We have that

$$\begin{aligned}
\tau\Big\{\chi_{[r,\infty)}(X_b)\,|\lambda(e^{-t\psi})|^2\Big\} &= \sup_{0<\varepsilon\leq R<\infty} \tau\Big\{\operatorname*{SOT-lim}_{n\to\infty} \chi_{r,n}(\mathcal{J}(\phi_{\varepsilon,R}))\,|\lambda(e^{-t\psi})|^2\Big\} \\
&= \sup_{0<\varepsilon\leq R<\infty} \lim_{n\to\infty} \tau\Big\{\mathcal{J}(\chi_{r,n}\circ\phi_{\varepsilon,R})\,|\lambda(e^{-t\psi})|^2\Big\} \\
&\leq \lim_{n\to\infty} \sup_{0<\varepsilon\leq R<\infty} \tau\Big\{\mathcal{J}(\chi_{r,n}\circ\phi_{\varepsilon,R})\,|\lambda(e^{-t\psi})|^2\Big\}.
\end{aligned}$$

Moreover, $\lambda(e^{-t\psi}) = \mathcal{J}(h_t)$ for the heat kernel h_t in $\mathbb{R}^n$ and

$$\begin{aligned}
\tau\Big\{\chi_{[r,\infty)}(X_b)\,|\lambda(e^{-t\psi})|^2\Big\} &= \lim_{n\to\infty} \sup_{0<\varepsilon\leq R<\infty} \tau\Big\{\mathcal{J}(\chi_{r,n}\circ\phi_{\varepsilon,R})\,|\mathcal{J}(h_t)|^2\Big\} \\
&= \lim_{n\to\infty} \sup_{0<\varepsilon\leq R<\infty} \tau\Big\{\mathcal{J}\big((\chi_{r,n}\circ\phi_{\varepsilon,R})\,|h_t|^2\big)\Big\} \\
&= \lim_{n\to\infty} \sup_{0<\varepsilon\leq R<\infty} \int_{\mathbb{R}^n} \chi_{r,n}(\phi_{\varepsilon,R}(\xi))\,|h_t(\xi)|^2\,d\xi \\
&\leq \lim_{n\to\infty} \int_{\mathbb{R}^n} \chi_{r,n}(|\xi|)\,|h_t(\xi)|^2\,d\xi \\
&= \int_{\mathbb{R}^n} \chi_{[r,\infty)}(|\xi|)\,|h_t(\xi)|^2 d\xi \\
&\lesssim \frac{1}{\Phi(\sqrt{t})} e^{-\frac{r^2}{2t}}.
\end{aligned}$$

The CBHL inequality will follow from the following L_∞ *Gaussian lower bounds*

$$\Big\|\Big(\chi_{[0,r)}(X_b)\,\lambda(e^{-t\psi})\,\chi_{[0,r)}(X_b)\Big)^{-1}\Big\|_{\mathcal{L}G}^{-1} \gtrsim \frac{e^{-\beta\frac{r^2}{t}}}{\Phi(\sqrt{t})}. \tag{L_∞GLB}$$

Recall that if $x \in \mathcal{M}_+$ and p is a projection then $p\|(pxp)^{-1}\|^{-1} \leq pxp$ and so we can understand the right hand side of (L_∞GLB) as a lower bound on $\chi_{[0,r)}(X_b)\,\lambda(e^{-t\psi})\,\chi_{[0,r)}(X_b)$. The L_∞GLB allow to bound the noncommutative Hardy-Littlewood maximal operator by the maximal operator associated with the semigroup. Indeed, since X_b and $\lambda(e^{-t\psi})$ commute from (3.4.1) we deduce that (L_∞GLB) yield

$$\frac{\chi_{[0,t)}(X_b)}{\Phi(t)} \lesssim \chi_{[0,t)}(X_b)\,\lambda(e^{-t^2\psi})\,\chi_{[0,t)}(X_b) \leq \lambda(e^{-t^2\psi}).$$

This implies

$$\frac{\chi_{[0,t)}(X_b)}{\Phi(t)} \star x \lesssim S_{t^2}(x),$$

for every positive x. Now, using the maximal inequalities for semigroups of [JX07] gives the boundedness of the noncommutative Hardy-Littlewood maximal for every $1 < p < \infty$. The fact that $S_t \otimes \mathrm{Id}$ is again a Markovian semigroup gives the complete bounds and so the CBHL inequality holds. To prove that (L_∞GLB) holds we use that $\mathcal{J} : \mathcal{A} \to \mathcal{L}G$ is a complete contraction. Justifying the calculations like in the case of upper L_2 Gaussian bounds and using (3.4.1) we obtain that

$$\begin{aligned}
\Big\| \Big(\chi_{[0,r)}(X_b)\lambda(e^{-t\psi})\chi_{[0,r)}(X_b) \Big)^{-1} \Big\|_\infty &= \big\| \lambda(e^{-t\psi})^{-\frac{1}{2}}\, \chi_{[0,r]}(X_b)\, \lambda(e^{-t\psi})^{-\frac{1}{2}} \big\|_\infty \\
&\leq \big\| \lambda(e^{-t\psi})^{-\frac{1}{2}}\, \widetilde{\chi}_r(X_b)\, \lambda(e^{-t\psi})^{-\frac{1}{2}} \big\|_\infty \\
&= \big\| \mathcal{J}\left(\widetilde{\chi}_r\, h_t^{-1} \right) \big\|_\infty \\
&= \big\| \widetilde{\chi}_r\, h_t^{-1} \big\|_{L_\infty(\mathcal{R}^n)} \quad \lesssim \quad t^{\frac{n}{2}} e^{\beta\, \frac{r^2}{t}}, \qquad (3.4.4)
\end{aligned}$$

where $\chi_{[0,r]}(|x|) \leq \widetilde{\chi}_r$ is a radial and analytic function approximating $\chi_{[0,r]}(|x|)$ such that $h_t^{-1}\,\widetilde{\chi}_r \lesssim t^{\frac{n}{2}}\, e^{\beta\,\frac{r^2}{t}}$. One can safely take $\widetilde{\chi}_r(x) = F(M(1-|x/r|^2))F(M(1+|x/r|^2))$ for large enough M, where F is defined like in (3.4.3). The analyticity is used to ensure that $\widetilde{\chi}_r(\mathcal{J}(\phi)) = \mathcal{J}(\widetilde{\chi}_r(\phi))$. In Line (3.4.4) we have used point (ii) in Lemma 3.4.2. □

Corollary 3.4.4. *Given a LCH amenable unimodular group G, let $b : G \to \mathbb{R}^n$ be a finite-dimensional cocycle with associated c.n. length $\psi(g) = |b(g)|^2$. Then, given a symbol $m : \mathbb{R}_+ \to \mathbb{C}$ and $1 < p < \infty$, the following estimate holds for any $\mathcal{H}_0^\infty$ cut-off function η and any $s > n/2$*

$$\big\| T_{m\circ\psi} \big\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t>0} \big\| m(t\,\cdot)\eta(\cdot) \big\|_{W^{2,s}(\mathbb{R}_+)}.$$

Proof. If the cocycle b is surjective and proper the result follows from Theorem 3.2.1. Indeed, in that case we know from Theorem 3.4.3 that (G, ψ, X_b) satisfies the standard assumptions with $\Phi(s) = s^n$ and Sobolev dimension $D_\Phi = n$. Moreover, the CBPlan_2^Φ property also holds as we explained before Lemma 3.4.1. In the general case take $G_\rtimes = \mathbb{R}^n \rtimes_\alpha G$ where $\alpha : G \to O(n)$ is the orthogonal representation that makes $g \mapsto (x \mapsto \alpha_g x + b(g))$ an affine representation. The function $b_\rtimes : G_\rtimes \to \mathbb{R}^n$ given by $b_\rtimes(\xi, g) = \xi + b(g)$ satisfies the cocycle law with cocycle action $\beta : G_\rtimes \to \mathbb{R}^n$ given by $\beta_{(\xi,g)} = \alpha_g$. Indeed, we have

$$\begin{aligned}
b_\rtimes(\xi + \alpha_g\zeta, g\,h) &= \xi + \alpha_g\zeta + b(gh) \\
&= \xi + \alpha_g\zeta + \alpha_g b(h) + b(g) \\
&= \beta_{(\xi,g)}(b_\rtimes(\zeta, h)) + b_\rtimes(\xi, g).
\end{aligned}$$

Furthermore $b_\rtimes$ is clearly surjective but it may not be proper. In that case, we shall take the associated affine representation $\pi_\rtimes : G_\rtimes \to \mathbb{R}^n \rtimes O(n)$ and note that the quotient representation $\pi_\rtimes^\circ : G_\rtimes^\circ = G_\rtimes / \ker(\pi_\rtimes) \to \mathbb{R}^n \rtimes O(n)$ satisfies that its associated cocycle $b_\rtimes^\circ : G_\rtimes^\circ \to \mathbb{R}^n$ is always proper (even if it is not injective). To see that, let $p_1 : \mathbb{R}^n \rtimes O(n) \to \mathbb{R}^n$ be the natural projection into the first component and consider a compact set $K \subset \mathbb{R}^n$. Then

$$(b^\circ_\rtimes)^{-1}[K] = (\pi^\circ_\rtimes)^{-1}[p_1^{-1}[K]] = (\pi^\circ_\rtimes)^{-1}[K \times O(n)]$$

and the last term is compact since $K \times O(n)$ is compact and $\pi^\circ_\rtimes$ is a continuous group isomorphism and hence proper. Summing up, we have the following commutative diagram

$$\begin{tikzcd}
& G \arrow[r, "b"] \arrow[d, hook] & \mathbb{R}^n \\
\mathbb{R}^n \rtimes_\alpha G = & G_\rtimes \arrow[ur, "b_\rtimes"] \arrow[d, two heads] & \\
(\mathbb{R}^n \rtimes_\alpha G)/\ker(\pi_\rtimes) = & G^\circ_\rtimes \arrow[uur, bend right, "b^\circ_\rtimes"'] &
\end{tikzcd}$$

According to Theorem 3.4.3, for the last cocycle we can use that $(G^\circ_\rtimes, \psi^\circ_\rtimes, X_{b^\circ_\rtimes})$ satisfy the standard assumptions, where $\psi^\circ_\rtimes$ is the c.n. length naturally associated to $b^\circ_\rtimes$. By Theorem 3.3.7, this implies

$$\big\| T_{m \circ \psi^\circ_\rtimes} \big\|_{\mathcal{CB}(L_p(\mathcal{L}G^\circ_\rtimes))} \lesssim_{(p)} \sup_{t>0} \big\| m(t\,\cdot)\eta(\cdot) \big\|_{W^{2,s}(\mathbb{R}_+)}.$$

Now, using de Leeuw's type periodization [CPPR15, Theorem 8.4 iii)] we obtain the same complete bounds for $T_{m\circ\psi_\rtimes}$ in $L_p(\mathcal{L}G_\rtimes)$ for every $1 < p < \infty$. In order to prove the assertion, we just need to restrict to the subgroup $\{0\} \times G \leq G_\rtimes$.This follows from the de Leeuw's restriction type result in [CPPR15, Theorem 8.4 i)]. □

3.5 Foreword

Here we are going to estate and discuss natural problems arising in the course of this Chapter.

Existence of standard triples.

Recall that, although the unbounded operator $X \in \mathcal{L}G_+^\wedge$ in the standard triple is required to prove Theorems 3.2.1 and 3.3.7, it does not show up in the statement of the theorem. The natural problem then is when, given ψ, we can construct X satisfying the standard assumptions. More precisely the problem is the following

Problem 3.5.1. Assume $\psi : G \to \mathbb{R}_+$ satisfies the CBR_Φ property, i.e:

$$\big\| S_t : L_2(\mathcal{L}G) \to \mathcal{L}G \big\|_{\mathrm{cb}} \lesssim \frac{1}{\Phi(\sqrt{t})^{\frac{1}{2}}} \tag{CBR_Φ}$$

Under which conditions exists $X \in (\mathcal{L}G)_+^\wedge$ such that (G, X, ψ) is standard?

Even giving a positive answer to the problem above in the case in which ψ is given by a general, non-surjective, 1-cocycle $b : G \to \mathbb{R}^n$ seems a challenging problem. Its technical difficulty arises from the necessity of doing Fourier analysis inside a hypersurface of $\mathbb{R}^n$. We conjecture the following.

Conjecture 3.5.2. If ψ satisfies CBR_Φ, for doubling Φ and comes from a finite-dimensional cocycle, then there is an X such that (G, X, ψ) is standard.

Our intuition is that, since maximal ergodic inequalities hold for noncommutative Markovian semigroups, the critical point to prove are the L_2GB. Indeed, even relaxing the Gaussian bounds to more general off-diagonal results seems challenging. There are plenty of examples of infinite dimensional 1-cocycles $b : G \to H$ with an associated $\psi(g) = \|b(g)\|_H^2$ satisfying CBR_Φ. But, so far, we haven't been able to produce examples of infinite dimensional cocycles with a positive solution to 3.5.1.

Problem 3.5.3. Are there infinite-dimensional cocycles whose associated ψ is part of a standard triple?

Either an example of such object or the proof that the L_2GB imply finite-dimensionality of the associated cocycle will be an extremely interesting result.

Locallity, finite speed of propagation and Gaussian Bounds.

In the works of [Sik96], [Sik04] an equivalence, with some extra assumptions, between the following three properties is studied

1. Gaussian bounds for a semigroup $S_t = e^{-t\,A}$, with respect to its natural gradient metric, see [SC09].

2. Finite speed of propagation of the solution of the wave equation associated to A, given by $\partial_{tt}^2 \, x_t + A \, x_t = 0$.

3. Locality of the infinitesimal generator A, see [SC09, pp. 313].

The equivalence above explains why the locality of A, added to Sobolev-type inequalities and the doubling condition, is enough to give a spectral Hörmander-Mikhlin multiplier result in the abelian case, see [SC09, Theorem 3.5].

The study of the properties above in the case of group algebras was started with the hope of manufacturing a noncommutative metric $X \in (\mathcal{L}G)_+^\wedge$ generalizing the wave-travelling distance. Nevertheless, the path seems more technical than we originally expected. Indeed, in order to understand such equivalence above in the noncommutative setting we started to study the framework of W^*-metrics introduced by Weaver and Kuperberg [KW12]. In Chapter 5 we study such relation and arrive, during the discussion in the Foreword of that chapter, to a way of defining off-diagonal bounds and finite speed of propagation from a W^*-metric. Locality can be similarly defined provided there is a W^*-metric. We already have partial results concerning the equivalence between finite speed of propagation and Gaussian bounds to be included in a forthcoming work [GP16b].

C_c^∞-cut-off functions and Hilbert-valued extensions.

In the statement of the classical Hörmander-Mikhlin Theorem the condition includes a compactly supported cut-off function $\eta \in C_c^\infty(\mathbb{R}_+)$ instead of an analytic function $\eta \in \mathcal{H}_0^\infty$. The main obstacle to prove our results with $\eta \in C_c^\infty(\mathbb{R}_+)$ is that the square function estimates of [JLMX06] need analyticity. The underlying reason is that they are dimension-independent estimates and the classical Littlewood-Paley estimates with compactly supported localization need finite dimensionality. But, since our standard hypotheses imply finite dimensionality it makes perfect sense trying to extend Theorems 3.2.1 and 3.3.7 to Hilbert-valued functions to prove a compactly supported square function estimate and then apply a bootstrap argument.

Conjecture 3.5.4. Let (G, ψ, X) be a standard triple, $\eta \in \mathcal{H}_0^\infty$, $s > (D_\Phi + 1)/2$ and $m : \mathbb{R}_+ \to H$ be a Hilbert-valued function we have

$$\big\| T_m^{[rc]} \big\|_{\mathcal{CB}(L_p(\mathcal{L}G), L_p(\mathcal{L}G; H^{r\,c}))} \lesssim_{(p)} \sup_{t \geq 0} \| \eta(\cdot)\, m(t\,\cdot) \|_{W^{2,s}(\mathbb{R}_+, H)} \quad \forall\, 1 < p < \infty,$$

where $T_m^{[rc]}$ is the Hilbert-valued multiplier given by extension of $\lambda_g \mapsto \lambda_g \otimes m(g)$.

In order to prove such conjecture, the natural thing to do will be to apply arguments like those used by T. Mei and J. Parcet in [MP09] in L_∞ and BMO. That should allow us to prove an operator-valued extension and then restrictions to the column or row would give the desired result. Unfortunately, technical obstacles appear in our semigroup-based case, since using the formulas on [JM12] for expressing the BMO-norm does not yield a clear candidate to maximal function.

Nevertheless, if the conjecture above holds it will be enough to prove that, for $\varrho \in C_c^\infty(\mathbb{R}_+)$, the Hilbert-valued multiplier $m : G \to L_2(\mathbb{R}_+, ds/s)$ given by

$$m(g)(s) = \varrho(s\, \psi(g)),$$

satisfies the condition in Conjecture 3.5.4. We would obtain the following as a Corollary.

Conjecture 3.5.5. If (G, ψ, X) is standard, then, the following hold

$$\|x\|_{L_p^\circ(\mathcal{L}G)} \sim_{(p)} \big\| (T_{\varrho(t\,\psi)} x)_{t \geq 0} \big\|_{L_p(\mathcal{L}G; L_2^{r\,c}(\mathbb{R}_+, \frac{dt}{t}))}.$$

By a bootstrap argument, repeating the steps in the proof of Theorems 3.2.1 and 3.3.7 we can replace the cut-off function in $\mathcal{H}_0^\infty$ by an element in $C_c^\infty(\mathbb{R}_+)$.

Another approach that may seem reasonable to prove Conjecture 3.5.5 is to use an operator-valued extension of Theorem 2.2.8, which is easily shown, to avoid working with maximals. Sadly that multiplier result does not yield the desired estimates since the square function $m(g) \mapsto (\varrho(s\,\psi(g)))_{s \geq 0}$ do not satisfies, in general, the operator-valued analogue of the condition in Theorem 2.2.8.

General symbols over homogeneous groups.

In the condition for spectral Hörmander-Mikhlin multipliers we have dilations given by $m(t\,\psi)$. But, in the case of $\mathbb{R}^n$, the Hörmander-Mikhlin condition admits more general nonradial symbols $m : \mathbb{R}^n \to \mathbb{C}$ and the dilations of the symbol are implemented by the group automorphisms $\xi \mapsto t\,\xi$. The natural setting for a generalization, if we want to obtain a nonradial Hörmander-Mikhlin multiplier result in $L_p(\mathcal{L}G)$, is to work with the so called *homogeneous groups*, see [Goo14]. A connected and locally connected Lie group G is said to be homogeneous iff there is a 1-parameter family of dilations $\delta : \mathbb{R}_+^* \to \mathrm{Aut}(G)$ given at Lie algebra level by

$$d\delta_t(X_j) = t^{n_j}\, X_j \text{ for some base } \{X_j\}_j \text{ of } T_e G.$$

Such groups are nilpotent and include important examples like the Heisenberg groups. Result of the form

$$\|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))} \lesssim_{(p)} \sup_{t\geq 0} \|\eta(\cdot)\, m(\delta_t \cdot)\|_{W^{2,s}(G)} \tag{3.5.1}$$

are very much expected to hold for $s > D(\delta)/2$, where $D(\delta)$ represents the degree of δ, given by $D(\delta) = \sum_j n_j$, and coinciding with the local dimension in many natural contexts. The harmonic analysis of homogeneous groups have been intensively studied in the past, see [FS82]. The approach required here have to be the dual one, instead of studying translation invariant operators in $L_p(G)$, we want to study comultiplication invariant operators in $L_p(\mathcal{L}G)$. It is also worth recalling that the ingredients used in our proof, like square function estimates, have not been developed yet in this context.

Is there a unified frame including hyperbolic groups?

The condition CBR_Φ is quite demanding for a group. Indeed, up to quotients by $G_0 = \{g \in G : \psi(g) = 0\} \leq G$ and restrictions to the support of $e^{-t\,\psi}$ the group G is amenable. Therefore, an important point that need clarification is the necessity of the complete boundedness on CBR_Φ —as opposed to plain boundedness $\mathrm{R}_\Phi^{p,q}$ —. The question is of outstanding importance. For instance, the study of bounded ultracontractivity on non-amenable groups was pioneered by Haagerup [Haa79] in the case of $\mathbb{F}_2$ and later generalized, under the name of *rapid decay (RD) property*, by Joliessaint [Jol90], see also [dlH88] for similar results in general hyperbolic groups. In our proof the complete bounds in CBR_Φ are necessary since we use in the proof of Theorem 2.1.1 the following inequalities

$$\begin{aligned} |T_{k_1\, k_2}(x)|^2 &\leq \|k_1\|_{L_\infty(\mathcal{M};L_2^r(\tau))}\, (\tau \otimes \mathrm{Id})\big\{k_2^* k_2\, (\sigma(x^* x) \otimes \mathbf{1})\big\} \\ &= \|T_{k_1}\|_{\mathcal{CB}(L_2(\tau),\mathcal{M})}\, T_{k_2^* k_2}(x^* x), \end{aligned} \tag{3.5.2}$$

where, $k \in \mathcal{L}G \,\overline{\otimes}\, \mathcal{L}G_{\mathrm{op}}$ is a kernel and and T_k is its associated integral operator given by $T_k(x) = (\tau \otimes \mathrm{Id})\{k\,(\sigma(x) \otimes \mathbf{1})\}$. In order to prove our result without complete boundedness we need to change the $\mathcal{CB}(L_2^r(\mathcal{L}G), \mathcal{L}G)$-norm in (3.5.2) by the $\mathcal{B}(L_2^r(\mathcal{L}G), \mathcal{L}G)$-norm which does not seems to be possible in general. Nevertheless, even if our techniques does not generalize to the non-completely bounded case, some smooth Fourier multiplier results seem to hold in the case of nonamenable hyperbolic groups like $\mathcal{L}\mathbb{F}_2$. Indeed, recent results of M. de la Salle and T. Mei, see [dlSM14], yield that

Theorem 3.5.6 ([dlSM14, Theorems 1.1/1.2]). *Let $\psi : \mathbb{F}_2 \to \mathbb{R}_+$ be the word-length function in $\mathbb{F}_2$. If $0 < \alpha \leq 1/2$ we have*

$$\|T_{\phi(\psi)}\|_{\mathcal{CB}(L_1(\mathcal{L}\mathbb{F}_2))} \lesssim_{(\alpha)} \big\|x^{\frac{3}{4}-\alpha}\, \partial_{xx}^2 \phi\big\|_{L_2(\mathbb{R}_+)}^{\frac{1}{2}} \big\|x^{\frac{3}{4}+\alpha}\, \partial_{xx}^2 \phi\big\|_{L_2(\mathbb{R}_+)}^{\frac{1}{2}},$$

for any $\phi : \mathbb{R}_+ \to \mathbb{C}$.

Applying their techniques in the case of Bochner-Riesz means gives that such means are bounded in L_1 for $\delta > 1$, see [dlSM14, Example 3.4]. In the classical case of $\mathbb{R}^D$ the are L_1-bounded when $\delta > (D-1)/2$. This suggests that the “dimension” of $\mathcal{L}\mathbb{F}_2$ is indeed 3, which coincides with the dimension we obtain in the non-c.b. version of $\mathrm{R}_\Phi^{p,q}$. We can formulate the following conjecture

Conjecture 3.5.7. An spectral Hörmander-Mikhlin theorem, analogous to Theorems 3.2.1 and 3.3.7, holds in $\mathcal{L}\mathbb{F}_2$ for smoothness $s > 3/2$.

A possible direction to this conjecture may be working with the semigroup of c.b. multipliers given by $\lambda_g \mapsto e^{-t\,|g|^r}\,\lambda_g$. The main obstacles of this approach are the loss of positivity and the loss of doublingness. Generalizing results on the harmonic analysis for nondoubling settings seems necessary.

Chapter 4

Crossed Product Extensions of c.b Operators

The purpose of this chapter is to study transference results for operators on the L_p-spaces of crossed products. In Section 4.2, we will see that if T_m is some Fourier multiplier, then its crossed product extension $\mathrm{Id} \rtimes T_m$ is completely bounded in $L_p(\mathcal{M} \rtimes_\theta G)$ whenever T_m is completely bounded over $L_p(\mathcal{L}G)$, provided that the action θ is *amenable*, see [BO08, Section 4.3] for a precise definition. Furthermore, we have that

$$\|\mathrm{Id} \rtimes T_m\|_{\mathcal{CB}(L_p(\mathcal{M} \rtimes_\theta G))} \leq \|T_m\|_{\mathcal{CB}(L_p(\mathcal{L}G))}.$$

The techniques involved in the proof of such result are a generalization of the theorems in [NR11] and [CdlS15] from amenable groups to amenable actions. One of the novelties of our approach is that it allows us to transfer, not just Fourier multipliers acting on the G-component of $\mathcal{M} \rtimes_\theta G$, but equivariant operators acting of $\mathcal{M}$. Indeed, strengthening the amenability of θ by imposing an accretivity condition on its generalized "Følner sets" gives a transference result for any completely bounded and θ-equivariant operator S in $\mathcal{CB}(L_p(\mathcal{M}))$ as follows

$$\|S \rtimes \mathrm{Id}\|_{\mathcal{CB}(L_p(\mathcal{M} \rtimes_\theta G))} \leq C^{\frac{1}{p}} \, \|S\|_{\mathcal{CB}(L_p(\mathcal{M}))}, \tag{4.0.1}$$

where $C > 1$ is a constant measuring the accretivity of such sets. In all examples of amenable actions we have worked down so far we can build approximating sequences whose accretivity is $C = 1$. We conjecture that such is the case for all amenable actions. In Section 4.1 we will state precisely the amenability condition required for our theorems and review briefly the equivalent definitions of amenability for actions.

In Section 4.3 we prove an operator-valued extension of the transference results described above. This allow us to prove *strong-type maximal inequalities* in crossed products. Concretely, if $(T_n)_{n \geq 0}$ is a family of completely positive Fourier multipliers over $L_p(\mathcal{L}G)$ and $(S_m)_{m \geq 0}$ is a family of completely positive and θ-equivariant operators in $L_p(\mathcal{M})$ we have an inequality of the form

$$\begin{aligned} & \Big\| {\sup_m}^+ {\sup_n}^+ \big\{ (S_m \rtimes T_n)(u) \big\} \Big\|_{L_p(\mathcal{M} \rtimes_\theta G)} \qquad (4.0.2) \\ \lesssim^{(\mathrm{cb})} \;\; & C^{\frac{1}{p}} \, \big\| (T_m)_m : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G; \ell_\infty) \big\|_{\mathrm{cb}} \, \big\| (S_n)_n : L_p(\mathcal{M}) \to L_p(\mathcal{M}; \ell_\infty) \big\|_{\mathrm{cb}} \, \|u\|_p, \end{aligned}$$

whenever θ has an approximating sequence with accretivity constant C. Observe that such inequality is a trivial consequence of Fubini type argument when $\mathcal{M}$ and G are abelian and the action θ is trivial. so that $\mathcal{M} \rtimes_\theta G = \mathcal{M} \,\overline{\otimes}\, \mathcal{L}G$. In Section 4.3 we will also obtain a result analogous to (4.0.2) but with $m = n$ and formulate our theorems for suprema over general, not necessarily discrete, index sets.

As a consequence of those maximal inequalities we obtain that the CBHL_p inequalities in Definition 3.1.1 are stable under crossed products if natural invariance conditions are satisfied. Since the rest of the standard assumptions are easily verified for crossed products, we obtain that the standard assumptions are stable under crossed products, see Theorem 4.3.1. Observe that, for this application, we can just use the amenability of G since, by Remark 1.10.1, the standard assumptions imply amenability.

The results exposed here are part of an ongoing research project that will be released for publication after finishing this PhD thesis, see [GP16a]. Apart from the material here presented we plan to obtain, as an application, new multiplier results in $L_p(\mathcal{L}G)$. The technique that we will employ will be to embed $\mathcal{L}G$ into $L_\infty(\Omega) \rtimes_\theta G$, through a Ω-valued multiplicative cocycle $\beta : G \to L_\infty(\Omega)$, then use (4.0.1) to obtain bounds for multipliers from bounds for equivariant operators in Ω, see Section 4.4 for more details.

4.1 Amenability of actions

The purpose of this section is to recall a few facts from the theory of amenable actions and provide suitable references.

Let (X, Σ_X, ν), or simply (X, ν) if the σ-algebra is understood from the context, be a σ-finite measure space. We will say that a group homomorphism $\theta : G \to \mathrm{Aut}(X, \nu)$ is action of G on X iff the map $(g, x) \mapsto \theta_g(x) = g\,x$ is measurable and $\theta_g^* \nu$ and ν are mutually absolutely continuous. When $\theta_g^* \nu(E) = \nu(\theta_g E) = \mu(E)$ we will say that θ is ν-preserving. The action θ extends trivially to an action over the functions on X. We are going to denote it, perhaps ambiguously, by $\theta_g f(x) = f(\theta_{g^{-1}} x)$. As usual, if there is no confusion we may just write $f(g^{-1}\,x)$ or $g\,f$ instead of $f(\theta_{g^{-1}} x)$.

Recall that a group G is said to be amenable if there is a translation invariant mean $m \in L_\infty(G)^*$, i.e. an element $m \in L_\infty(G)^*$ such that $m(f) \geq 0$ for every $f \geq 0$, $m(\chi_G) = 1$ and $m(f(g^{-1} \cdot)) = m(f)$ for every $g \in G$. We now define a weaker notion of amenability for an action on a von Neumann algebra.

Definition 4.1.1. Let $\theta : G \to \mathrm{Aut}(X, \nu)$ be an action, we will say that it is amenable iff there is a (not necessarily normal) θ-equivariant conditional expectation $\mathcal{E} : L_\infty(X) \,\overline{\otimes}\, L_\infty(G) \to L_\infty(X)$, i.e. a unital, positivity preserving and $L_\infty(X)$-linear map, such that

$$\mathcal{E}\, f(\theta_{g^{-1}} \cdot, g^{-1} \cdot) = \theta_g\, \mathcal{E}(f).$$

If $(\mathcal{M}, \varphi)$ is a von Nuemann algebra and φ a normal, semifinite and faithful weight, an action $\theta : G \to \mathrm{Aut}(\mathcal{M})$ is amenable iff its restriction to the abelian subalgebra $(\mathcal{Z}(\mathcal{M}), \varphi|_{\mathcal{Z}(\mathcal{M})})$ is amenable, where $\mathcal{Z}(\mathcal{M})$ denotes the center of $\mathcal{M}$.

Observe that, trivially, if G is amenable all of its actions are amenable, just take $\mathcal{E} = \mathrm{Id}_{L_\infty(X)} \otimes m$, for any G-invariant mean m. Reciprocally if G acts amenably on a one-point space then G is amenable. The flexibility gained is that non-amenable groups may have nontrivial amenable actions. We may also recall that if G acts amenably in a probability space (X, ν) and the measure

ν is invariant, then, the composition $\nu \circ \mathcal{E}$ is an invariant mean. The same holds for finite measure spaces with a θ-invariant measure.

Like in the case of amenability there are several equivalent characterizations of the property. The definition we have introduced is not the one that appeared first in the literature. The oldest one, to the knowledge of the author, is that an action $\theta : G \to \mathrm{Aut}(X, \nu)$ is amenable iff every affine action on a weak-$*$ compact convex set *subordinated to* θ has a fixed point. A weak-$*$ compact convex G-set $K \subset E^*$ is said to be subbordinated to $\theta : G \to \mathrm{Aut}(X, \nu)$ iff E^* can be constructed by tensoring $L_\infty(X)$ with some dual space E_0^* and twisting with a 1-cocycle $\alpha : G \to \mathfrak{B}(X, \mathrm{Iso}(E_0))$, where $\mathfrak{B}(X, A)$ are the Borelian functions. A very detailed introduction to such concept can be found in [Zim84, Chapter 4]. Of course, when $X = \{p\}$, we get that any affine action of G in a compact weak-$*$ closed subset has a fixed point, a condition long known to be equivalent to amenability, see [Pat88]. Amenable action were introduced in the pioneering works of Zimmer, see [Zim77],[Zim78a], [Zim78b],[Zim78c] following earlier results of Furstenberg [Fur73]. The equivalence with the definition here given was proved in [AEG94].

It is important to recall that amenability of actions can be defined for continuous actions on topological spaces. Pretty much in the same way in which measurable groups are somehow the same objects as topological groups, see [Var85a, Chapter 5:6], topological amenable actions are the same object as Borel amenable actions. In order to clarify this we will need the following proposition. Recall that we are going to denote by $\mathbb{P}(G)$, the probability measures with the $\sigma(C_0(G))$-topology and by $\mathbb{P}_0(G)$ the subset of all absolutely continuous ones with respect to the Haar measure.

Proposition 4.1.1. *Let $\theta : G \to \mathrm{Aut}(X, \nu)$ be an action, it is amenable iff for every $m \in \mathbb{P}_0(G)$, $\epsilon > 0$ and $K \subset G$ a compact subset there is a Borel map $\mu : X \to \mathbb{P}_0(G)$ such that*

$$\sup_{g \in K} \int_X \| g\, \mu^x - \mu^{g\, x} \|_1 \, d\, m(x) < \epsilon, \tag{4.1.1}$$

where $g\, d\, \mu^x(h) = d\, \mu^x(g^{-1}\, h)$.

Whenever a net $(\mu_\alpha)_\alpha$ satisfies condition (4.1.1) for every m, ϵ and K provided that α is large we will say that $(\mu_\alpha)_\alpha$ is *asymptotically equivariant.* Observe that the condition in the proposition above is equivalent to the existence of an asymptotically equivariant net. To see that, just denote by $\mu_{m,\epsilon,K}$ the Borel measurable map in Proposition 4.1.1 and by A the net given by all triples (m, ϵ, K) with the natural order.

Proof. Given any Borel map $\mu : X \to \mathbb{P}_0(G)$ we can associate to it a unital, positivity preserving and $L_\infty(X)$-linear map $\mathcal{E}_\mu : L_\infty(X) \,\overline{\otimes}\, L_\infty(G) \to L_\infty(X)$ given by

$$\mathcal{E}_\mu(f)(x) = \int_G f(x, g)\, d\, \mu^x(g).$$

Clearly all such maps have norm bounded by $\|\mathcal{E}_\mu(\mathbf{1})\|_\infty = 1$. The space of bounded maps $\mathcal{B}(L_\infty(X \times G), L_\infty(X))$ is a dual Banach space since

$$\begin{array}{rcl} \mathcal{B}(L_\infty(X \times G), L_\infty(X)) & = & L_\infty(X) \,\overline{\otimes}\, L_\infty(X \times G)^* \\ & = & L_1(X)^* \,\overline{\otimes}\, L_\infty(X \times G)^* \\ & = & (L_1(X) \,\widehat{\otimes}\, L_\infty(X \times G))^* \quad = \quad L_1(X; L_\infty(X \times G))^*, \end{array}$$

and the pairing is given by extension of

$$\langle m \otimes f, \mathcal{E} \rangle = \langle m, \mathcal{E}(f) \rangle = \int_X \mathcal{E}(f)\, d\, m.$$

Therefore, by the Banach-Alaoglu compactness theorem, the net $(\mathcal{E}_{\mu_\alpha})_\alpha$ has a weak-$*$ accumulation point $\mathcal{E}$. Since the subset of all conditional expectations is clearly weak-$*$ closed, $\mathcal{E}$ is also a conditional expectation. We have to see that if $(\mu_\alpha)_\alpha$ is asymptotically invariant, then $\mathcal{E}$ is equivariant. But that is obvious since we have

$$\begin{aligned}\langle \mathcal{E}_{\mu_\alpha}(\theta_g f) - \theta_g(\mathcal{E}_{\mu_\alpha}(f), m\rangle &= \int_X \int_G f(g^{-1}\,x, g^{-1}\,h)\left\{d\,\mu_\alpha^x(g\,h) - d\,\mu_\alpha^{g^{-1}}(h)\right\} d\,m(x) \\ &\leq \|f\|_\infty \int_X \left\|g^{-1}\,\mu_\alpha^x - \mu_\alpha^{g^{-1}\,x}\right\|_1 d\,m(x)\end{aligned}$$

and for every $g \in G$ such quantity can be made arbitrarily small.

For the reciprocal we have to use that the space of normal conditional expectations from $L_\infty(X \times G)$ to $L_\infty(X)$ is dense inside the set of all conditional expectations with respect to the weak-$*$ topology. Notice that, by applying the Hahn-Banach theorem in every fibre, normal conditional expectations are in correspondence with measurable maps $\mu : X \to \mathbb{P}_0(G)$. If $\mathcal{E}$ is an equivariant conditional expectation we have that there is a net (μ_α) of Borel maps with $\mathcal{E}_{\mu_\alpha} \to \mathcal{E}$ in the weak-$*$ topology. The net μ_α is asymptotically equivariant. After identifying Borel maps $X \to \mathbb{P}_0(G)$ with a subset of $L_\infty(X; L_1 G)$ we have that the weak-$*$ topology of $\mathcal{B}(L_\infty(X \times G), L_\infty(X))$ corresponds to the $\sigma(L_1(X; L_\infty G))$ topology. In particular, for every $g \in G$ we have that $g\,\mu_\alpha^x - \mu_\alpha^{g\,x}$ tends to zero in the $\sigma(L_1(X; L_\infty G))$ topology. In particular 0 is in the $\sigma(L_1(X; L_\infty G))$-closed convex hull of the set of all the maps

$$S_g = \left\{x \mapsto (g\,\mu^x - \mu^{g\,x})\right\}.$$

It is easily seen that such convex set equals the closure of S_g in the coarsest linear topology making all maps

$$\mu \mapsto \int_X \|\mu^x\|_1\, d\,m(x)$$

continuous. Taking a sequence in the convex hull of S_g converging to 0 in such topology gives the claim. □

We recall now the definition of amenability for topological actions. We will say that an action of G by homeomorphisms $\theta : G \to \mathrm{Homeo}(X)$ is a topological action iff the map $(g, x) \mapsto \theta_g(x)$ is continuous.

Definition 4.1.2. Let X be a locally compact topological space and θ a topological action. The action is said to be amenable iff there is a net of continuous maps $\mu_\alpha : X \to \mathbb{P}(G)$, such that for every $g \in G$

$$\lim_\alpha \sup_{x \in X} \|g\,\mu_\alpha^x - \mu_\alpha^{g\,x}\|_1 = 0.$$

Similarly, an action of G in a C^*-algebra $\mathcal{A}$ is amenable iff its restriction to the center $\mathcal{Z}(\mathcal{A})$ is (topologically) amenable.

Observe that, since $\mathbb{P}(G)$ is a compact, each μ_α can be lifted to a continuous function on βX, its Stone-Čech compactification and so we obtain that, by construction, a continuous action θ on X is amenable iff its lift $\beta\theta$ to βX is amenable.

Such topological definition of amenability appeared in the form above for the first time in [HR00]. In contemporary literature is more common to see amenable actions defined in terms of topological spaces. The topic of topological amenable actions has been researched in connection with exactness for groups, a notion introduced in [KW99], since it was proved in [Oza00] that a discrete group is exact iff it has an amenable action on a compact space. See also [Oza06] for more on amenable actions.

Recall that we can identify continuous functions $x \in C_c(G;\mathcal{M})$ with elements inside $\mathcal{M} \rtimes_\theta G$ and that, whenever G is discrete, we have a natural conditional expectation given by $\mathbb{E}_\mathcal{M}(x) = (\mathrm{Id} \rtimes \tau_G)(x) = x(0)$. In the case of crossed products of von Neumann algebras we can easily extend $\mathbb{E}_\mathcal{M}$ when G is nondiscrete as an operator-valued weight $\mathbb{E}_\mathcal{M} : (\mathcal{M} \rtimes_\theta G)_+ \to \mathcal{M}_+^\wedge$, see [Haa79a],[Haa79b]. It is easy to see that

$$\mathbb{E}_\mathcal{M}\big[x\,x^*\big] = \int_G x(g)\,x(g)^*\,d\,\mu(g), \tag{4.1.2}$$

for any $x, y \in \mathcal{M} \rtimes_\theta G$. When working with C^* algebraic crossed products like bellow there is no ambiguity assuming $\mathcal{A} \subset \mathcal{A}^{**}$ to define $\mathbb{E}_\mathcal{A}$. The characterization below is easily seen to be equivalent to amenability.

Lemma 4.1.2. ([BO08, Definition 4.3.1/Lemma 4.3.7]) *An continuous action $\theta : G \to \mathrm{Aut}(\mathcal{A})$, where $\mathcal{A}$ is a unital C^*-algebra is amenable iff there is a net $(x_\alpha)_\alpha \subset C_c(G;\mathcal{Z}(\mathcal{A}))$ of compactly supported functions satisfying*

(i) $0 \leq x_\alpha(g)$.

(ii) $\displaystyle\int_G |x_\alpha(g)|^2\,d\,\mu(g) = \mathbf{1}_\mathcal{A}$.

(iii) $$\begin{aligned}\lim_\alpha \mathbb{E}_\mathcal{A}\big\{((\mathbf{1} \rtimes \lambda_g)\,x_\alpha - x_\alpha)\,((\mathbf{1} \rtimes \lambda_g)\,x_\alpha - x_\alpha)^*\big\} &= \lim_\alpha \int_G |\theta_g(x_\alpha(g^{-1}\,h)) - x_\alpha(h)|^2\,d\,\mu(h) \\ &= 0.\end{aligned}$$

Any such net will be called an approximating sequence*.*

The following proposition ensures that if X is the Borel space underlying a compact space and θ a continuous action, then θ is amenable in the measurable sense iff it is amenable in the topological sense.

Proposition 4.1.3 ([BO08, Proposition 5.2.1]). *Let X be a compact Hausdorff space and θ a continuous action of G on X. Then θ is (topologically) amenable iff we can take a net of asymptotically equivariant Borel maps $\mu_\alpha : X \to \mathbb{P}_0(G)$.*

It rests to see that any measurable action comes from a topological action.

Theorem 4.1.4 ([Var85a, Theorem 5.7]). *For any measurable action θ of G in X there is a compact Hausdorff topological space Y, a continuous action θ_0 of G on Y and a θ_0-invariant Borel subset $E \subset Y$ such that X and E are isomorphic as G-spaces.*

Sometimes the Borel subset E above can be taken to be closed without loss of generality. One of such situations is when the action preserves a finite measure. Let (X,ν) be a finite measure space and the action θ of G be ν-preserving. If $E \subset Y$ is like in the theorem above and $\iota : X \hookrightarrow E \subset Y$ we have that the finite measure $\iota_*\nu \in M(Y)$ is Borelian and its support is a closed G-invariant subset $\mathrm{supp}[\iota_*\nu] \subset E$. Restricting to such support amounts to removing a null set of X, see [AEG94, Lemma 1.3]. Similar results follow for ν-preserving actions when ν is an infinite regular measure changing closed sets by locally closed sets.

As a corollary of the following discussion we obtain that any action $\theta : G \to \mathrm{Aut}(\mathcal{M})$ of G on a von Neumann algebra $\mathcal{M}$ is amenable iff it has an approximating sequence $(x_\alpha)_\alpha \subset C_c(G;\mathcal{Z}(\mathcal{M}))$ as in Lemma 4.1.2. We introduce now the refinement of amenability of actions that we are going to use through the next subsections.

Definition 4.1.3. Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra and denote $(\mathcal{Z}(\mathcal{M}),\tau|_{\mathcal{Z}(\mathcal{M})})$ by $(L_\infty(X),\nu)$. We say that the action $\theta : G \to \mathrm{Aut}(\mathcal{M})$ has a C-approximating sequence iff there is a sequence of sets $F_\alpha \subset X \times G$ such that

$$1 \leq \operatorname*{ess\,sup}_x \mu\{g \in G : (x,g) \in F_\alpha\} \leq C \operatorname*{ess\,inf}_x \mu\{g \in G : (x,g) \in F_\alpha\} < \infty, \tag{4.1.3}$$

and the elements

$$x_\alpha(x,g) = \frac{\chi_{F_\alpha}(x,g)}{\mu\{g \in G : (x,g) \in F_\alpha\}^{\frac{1}{2}}}$$

form an approximating sequence satisfying (iii) in Lemma 4.1.2.

Clearly we have

$$G \text{ is amenable} \quad \Longrightarrow \quad \theta \text{ has a } C\text{-approximating sequence} \quad \Longrightarrow \quad \theta \text{ amenable.}$$

Many known amenable actions admit C-approximating sequences, Furthermore the existence of C-approximating sequences is stable under natural operations like tensor product extensions $\mathrm{Id} \otimes \theta : G \to \mathrm{Aut}(\mathcal{M} \overline{\otimes} \mathcal{M}_2)$. Diagonal products $\theta_1 \times \theta_2 : G \to \mathrm{Aut}(\mathcal{M}_1 \overline{\otimes} M_2)$ and tensor products $\theta_1 \otimes \theta_2 : G_1 \times G_2 \to \mathrm{Aut}(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)$.

Example 4.1.5. We are going to see that the free group $\mathbb{F}_n$ acting on its hyperbolic boundary $\partial\mathbb{F}_n$ is an amenable action admitting a 1-approximating sequence. Recall the construction of $\partial\mathbb{F}_n$. We want such space to be the set of all infinite reduced words

$$\partial\mathbb{F}_n = \{\omega = s_0^{\epsilon_0}\, s_1^{\epsilon_1}\, s_2^{\epsilon_2} ...\} \subset \prod_{i=0}^{\infty} \{a_1^{\pm}, a_2^{\pm}, ..., a_n^{\pm}\},$$

where $\{a_1, a_2, ..., a_n\}$ is the set of generators. Clearly the last space is compact Hausdorff and totally disconnected. Since the different relations $w_i \neq w_{i+1}^{-1}$ define closed subsets, $\partial\mathbb{F}_n$ is also a compact space. The action $\theta_{\omega_0}(\omega)$ of $\mathbb{F}_n$ is given by adjoining any word of $\omega_0 \in \mathbb{F}_n$ to the infinite word $\omega \in \partial\mathbb{F}_n$ and performing the due reductions. The condition on Definition 4.1.2 is easily verified for

$$x_n(\omega) = \frac{1}{\sqrt{n+1}} \sum_{i=0}^{n} \delta_{\omega_i} = \frac{\chi_{F_n}}{\sqrt{n+1}},$$

where $\omega = \omega_0\, \omega_1 ... \in \partial\mathbb{F}_n$ and the sets $F_n \subset X \times G$ satisfies that $(\omega_1, \omega_2) \in F_n$ iff ω_1 is an initial subword for ω_2 of word length less or equal to n. Similar means yield that action of hyperbolic groups in their hyperbolic boundary have 1-approximating sequences, see [BO08, Chapter 5].

4.2 An asymptotic embedding

In this section we are going to prove the main result of this chapter. Observe that if $\theta : G \to \mathrm{Aut}(\mathcal{M})$ is an action and $\mathcal{M} \rtimes_\theta G$ is the reduced or spatial crossed product, then, the embedding of

$\mathcal{M} \rtimes_\theta G$ into $\mathcal{B}(H \otimes_2 L_2 G)$ factors through the subalgebra $\mathcal{M} \overline{\otimes} \mathcal{B}(L_2)$. Indeed, after identifying kernels k in $L_\infty(G \times G; \mathcal{M})$ with operators in $\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$, the embedding $j : \mathcal{M} \rtimes_\theta G \to \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$ is given by extension of the map sending $u \in C_c(G; \mathcal{M})$ to the operator with kernel

$$k(g, h) = [\theta_g^{-1}(u(g\, h^{-1}))]_{g,h \in G}.$$

Let $T_m : \mathcal{L}G \to \mathcal{L}G$ be a normal and c.b. Fourier multiplier of symbol m and denote by $(\mathrm{Id} \rtimes T_m)$ its crossed product amplification, i.e. the normal operator given by linear extension of the map $x \rtimes \lambda_g \mapsto m(g)\, x \rtimes \lambda_g$. A trivial calculation show that the isometry j intertwines $\mathrm{Id} \rtimes T_m$ and $\mathrm{Id} \otimes M_m$ as shown below

$$\begin{tikzcd} \mathcal{M} \rtimes_\theta G \arrow[r, hook, "j"] \arrow[d, "\mathrm{Id} \rtimes T_m"] & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \arrow[d, "\mathrm{Id} \otimes M_m"] \\ \mathcal{M} \rtimes_\theta G \arrow[r, hook, "j"] & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G), \end{tikzcd}$$

where $M_m : \mathcal{B}(L_2 G) \to \mathcal{B}(L_2 G)$ is the c.b. Herz-Schur multiplier given by

$$M_m([a_{g\,h}]_{g,h}) = [m(g\, h^{-1})\, a_{g\,h}]_{g\,h}.$$

Similarly, let $S : \mathcal{M} \to \mathcal{M}$ be an operator and let us denote by $S \rtimes \mathrm{Id}$ its crossed product amplification, i.e. the map given by extension of $x \rtimes \lambda_g \mapsto S(x) \rtimes \lambda_g$. An straightforward calculation shows that the embedding j intertwines $S \rtimes \mathrm{Id}$ and S_θ as follows

$$\begin{tikzcd} \mathcal{M} \rtimes_\theta G \arrow[r, hook, "j"] \arrow[d, "S \rtimes \mathrm{Id}"] & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \arrow[d, "S_\theta"] \\ \mathcal{M} \rtimes_\theta G \arrow[r, hook, "j"] & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G), \end{tikzcd}$$

where the map $S_\theta : \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \to \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$ is given by

$$S_\theta([x_{g\,h}]) = [\theta_g^{-1}\, S\, \theta_g(x_{g\,h})]_{g,h \in G}.$$

Therefore, if $S : \mathcal{M} \to \mathcal{M}$ is a normal c.b. and θ-equivariant operator we obtain that

$$\begin{tikzcd} \mathcal{M} \rtimes_\theta G \arrow[r, hook, "j"] \arrow[d, "S \rtimes \mathrm{Id}"] & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G) \arrow[d, "S \otimes \mathrm{Id}"] \\ \mathcal{M} \rtimes_\theta G \arrow[r, hook, "j"] & \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G). \end{tikzcd}$$

Observe that, a posteriori, such intertwining identities imply that if M_m is completely bounded so is $\mathrm{Id} \rtimes T_m$ and that if $S : \mathcal{M} \to \mathcal{M}$ is completely bounded and θ-equivariant so is $S \rtimes \mathrm{Id}$. It is a well-known result, see [BF91], [CdlS15], that the c.b. norm of the Fourier multiplier T_m bounds the c.b. norm of the Herz-Schur multiplier M_m. Summing all up, we obtain the following inequalities

$$\begin{array}{rclcl} \|\mathrm{Id} \rtimes T_m\|_{\mathrm{cb}} & \leq & \|\mathrm{Id} \otimes M_m\|_{\mathrm{cb}} & \leq & \|T_m\|_{\mathrm{cb}} \\ \|S \rtimes \mathrm{Id}\|_{\mathrm{cb}} & \leq & \|S \otimes \mathrm{Id}\|_{\mathrm{cb}} & = & \|S\|_{\mathrm{cb}}. \end{array}$$

The purpose of this section is to generalize such results from the crossed product von Neumann algebra $\mathcal{M} \rtimes_\theta G$ to its noncommutative L_p-spaces. The main difficulty stems from the fact that the isometry j is not trace preserving. Recall that, if $(\mathcal{M}, \tau_\mathcal{M})$ is a semifinite von Neumann algebra with a n.s.f. trace, θ is $\tau_\mathcal{M}$-preserving and G is LCH and unimodular, then there is a n.s.f. trace

$\tau : (\mathcal{M} \rtimes_\theta G)_+ \to [0, \infty]$ given by $\tau = \tau_G \circ \mathbb{E}_\mathcal{M}$ which generalizes both τ_G over $\mathcal{L}G \subset \mathcal{M} \rtimes_\theta G$ and $\tau_\mathcal{M}$ over $\mathcal{M} \subset \mathcal{M} \rtimes_\theta G$. It is easy to see that if G is a finite group, we have that

$$(\tau_\mathcal{M} \otimes \mathrm{Tr})(j\,\mathbf{1}) = |G|\,\tau(\mathbf{1}).$$

Therefore j is unbounded in $L_1(\mathcal{M} \rtimes_\theta G)$ when G is discrete and infinite. Similar arguments yield that j is ill-defined in L_1 when G is noncompact. A way of circumvent this difficulty is to use amenability to approximate the map j over compact subsets of G. This way of proceeding was used by E. Ricard and S. Neuwirth in [NR11], when $\mathcal{M} = \mathbb{C}$ and G is a discrete amenable group, to prove that if a Herz-Schur multiplier is completely bounded in $S_p(L_2 G)$, then so is the Fourier multiplier with the same symbol in $L_p(\mathcal{L}G)$. Their result was generalized later by M. Caspers and M. de la Salle in [CdlS15] to LCH amenable groups. They also proved that amenability is necessary for such theorem, at least for $4 < p$. We are going to generalize such transference results from amenable groups to amenable actions and from the L_p-spaces of group algebras $\mathcal{L}G$ to the L_p-spaces of crossed products. The way by which we are going to proceed is to use amenability to approximate j by a net $j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G))$ of complete contractions such that they are "asymptotically isometric". Then, we can obtain a complete isometry by taking an ultraproduct of all such maps, getting

$$(j_\alpha)_\alpha^\mathcal{U} : L_p(\mathcal{M} \rtimes_\theta G) \longrightarrow \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G)).$$

Recall that the ultraproduct above must be understood in the operator space sense, see [ER00, Appendix].

Let us start proving the following lemma.

Lemma 4.2.1. *Let $(\mathcal{M}, \tau)$, $\theta : G \to \mathrm{Aut}(\mathcal{M})$ be as above and assume that θ is τ-preserving and amenable and that G is unimodular. Let $(x_\alpha)_\alpha \subset C_c(G; \mathcal{Z}(\mathcal{M}))$ be any approximating net for θ and $X_\alpha \in \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G)$ be*

$$(X_\alpha\, \xi)(g) = \theta_g^{-1}(x_\alpha(g))\, \xi(g),$$

where $\xi \in L_2(G; H)$ The maps $j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G))$ given by

$$j_p^\alpha(v) = X_\alpha^{\frac{1}{p}}\, j(v)\, X_\alpha^{\frac{1}{p}}$$

satisfy that

(i) $\left\| j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G)) \right\|_{\mathrm{cb}} \leq 1$, *for every* $1 \leq p \leq \infty$.

(ii) $\lim_\alpha \big\langle (j_p^\alpha\, u), (j_{p'}^\alpha\, v) \big\rangle = \langle u, v \rangle$, *where* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, *for every* $1 \leq p < \infty$.

Proof. The proof of (i) is trivial when $p = \infty$. Proving it for $p = 1$ and applying interpolation yields the desired result. Let $u \in L_1(\mathcal{M} \rtimes_\theta G)$. We can decompose u as $x = a\, b^*$, with $\|u\|_2 = \|v\|_2 = \|x\|^{\frac{1}{2}}$. We have that

$$j_1^\alpha(u) = X_\alpha\, j(a)\, j(b)^*\, X_\alpha = (X_\alpha\, j(u))\, (X_\alpha\, j(v))^*.$$

But, clearly

$$\|j_1^\alpha(x)\|_{L_1(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G))} \leq \|X_\alpha\, j(u)\|_{L_2(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G))}\, \|X_\alpha\, j(v)\|_{L_2(\mathcal{M} \mathbin{\overline{\otimes}} \mathcal{B}(L_2 G))}.$$

It is trivial to notice that, since τ in θ-invariant $L_2(\mathcal{M} \rtimes_\theta G) = L_2(\mathcal{M}) \otimes_2 L_2(G)$ and the isomorphism is given by

$$\langle u, v \rangle_{L_2(\mathcal{M} \rtimes_\theta G)} = \int_G \tau_{\mathcal{M}}(u(g)^* \, v(g)) \, d\,\mu(g),$$

after identifying u affiliated with $\mathcal{M} \rtimes_\theta G$ with an $\mathcal{M}$-valued function of G in the natural way. On the other hand, if we denote by $k(g,h) = \theta_g^{-1}(x_\alpha(g)) \, \theta_g^{-1}(u(g\,h^{-1}))$ the kernel of $X_\alpha \, j(u)$, we have that

$$\begin{aligned}
\|X_\alpha \, j(u)\|^2_{L_2(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G))} &= (\tau_{\mathcal{M}} \otimes \mathrm{Tr}) \Big\{ \Big[\int_G k(g,h)^* \, k(k,h) \, d\,\mu(h) \Big]_{g,k} \Big\} \\
&= \int_G \int_G \tau_{\mathcal{M}} \big\{ |\theta_g^{-1}(x_\alpha(g)) \, \theta_g^{-1}(u(g\,h^{-1}))|^2 \big\} \, d\,\mu(g) \, d\,\mu(h) \\
&= \int_G \int_G \tau_{\mathcal{M}} \big\{ |x_\alpha(g) \, u(h^{-1})|^2 \big\} \, d\,\mu(g) \, d\,\mu(h) \qquad (4.2.1) \\
&= \int_G \tau_{\mathcal{M}} \Big\{ \Big(\int_G |x_\alpha(g)|^2 \, d\,\mu(g) \Big) \, |u(h^{-1})|^2 \Big\} \, d\,\mu(h) \\
&= \int_G \tau_{\mathcal{M}} \big\{ |u(h)|^2 \big\} \, d\,\mu(h), \qquad (4.2.2)
\end{aligned}$$

by using the θ-invariance of $\tau_{\mathcal{M}}$ in (4.2.1) and Condition (ii) on Lemma 4.1.2 as well as the unimodularity of G in (4.2.1). The same follows for v and this proves (i).

In order to prove (ii) start by noticing that

$$\begin{aligned}
\langle j_p^\alpha(u), j_p^\alpha(v) \rangle &= \int_G \int_G \tau_{\mathcal{M}} \big\{ \theta_g^{-1}(x_\alpha(g)) \, \theta_g^{-1}(u(g\,h^{-1})^* \, v(g\,h^{-1})) \, \theta_h^{-1}(x_\alpha(h)) \big\} \, d\,\mu(g) \, d\,\mu(h) \\
&= \int_G \int_G \tau_{\mathcal{M}} \big\{ \theta_{g\,h}^{-1}(x_\alpha(g\,h)) \, \theta_{g\,h}^{-1}(u(g)^* \, v(g)) \, \theta_h^{-1}(x_\alpha(h)) \big\} \, d\,\mu(g) \, d\,\mu(h) \\
&= \int_G \int_G \tau_{\mathcal{M}} \big\{ x_\alpha(g\,h) \, \theta_g(x_\alpha(h)) \, u(g)^* \, v(g) \big\} \, d\,\mu(g) \, d\,\mu(h) \\
&= \int_G \tau_{\mathcal{M}} \{ u(g)^* \, v(g) \} \, d\,\mu(g) + \int_G \tau_{\mathcal{M}} \big\{ u(g)^* \, v(g) \, A \big\} \, d\mu(g),
\end{aligned}$$

where A is just

$$\begin{aligned}
A &= \int_G x_\alpha(g\,h) \, \theta_{g^{-1}}(x_\alpha(h)) \, d\mu(h) - \mathbf{1}_{\mathcal{M}} \\
&= \int_G x_\alpha(g\,h) \, \theta_{g^{-1}}(x_\alpha(h)) \, d\mu(h) - \int_G |x_\alpha(g)|^2 \, d\,\mu(g) \\
&= \int_G x_\alpha(h) \, \big(\theta_{g^{-1}}(x_\alpha(g\,h)) - x_\alpha(h) \big) \, d\mu(h) \\
&\leq \Big\| \int_G |x_\alpha(h)|^2 \, d\,\mu(h) \Big\|_{\mathcal{M}}^{\frac{1}{2}} \Big(\int_G |\theta_{g^{-1}}(x_\alpha(g\,h)) - x_\alpha(h)|^2 \, d\mu(h) \Big)^{\frac{1}{2}} \\
&= \mathbb{E}_{\mathcal{M}} \big[((\mathbf{1} \rtimes \lambda_{g^{-1}}) \, x_\alpha - x_\alpha) \, ((\mathbf{1} \rtimes \lambda_{g^{-1}}) \, x_\alpha - x_\alpha)^* \big]^{\frac{1}{2}} \longrightarrow 0,
\end{aligned}$$

notice that we have used identity (4.1.2) in the last step. Using Condition (iii) of Lemma 4.1.2 and the Dominated Convergence Theorem gives the desired claim. □

We can now proceed to prove the main theorem of this section.

Theorem 4.2.2. *Let $(\mathcal{M}, \tau_{\mathcal{M}})$, G and $\theta : G \to \mathrm{Aut}(\mathcal{M})$ be as above with θ amenable. For any $1 \leq p < \infty$ we have a completely positive and completely isometric map*

$$L_p(\mathcal{M} \rtimes_\theta G) \xrightarrow{\ j_p\ } \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)).$$

The isometry j_p satisfies that if M_m and T_m are the Fourier and Herz-Schur multipliers associated to the symbol m, then

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_\theta G) & \xrightarrow{\ j_p\ } & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)) \\ \Big\downarrow {\scriptstyle (\mathrm{Id} \rtimes T_m)} & & \Big\downarrow {\scriptstyle (\mathrm{Id} \otimes M_m)^{\mathcal{U}}} \\ L_p(\mathcal{M} \rtimes_\theta G) & \xrightarrow{\ j_p\ } & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)), \end{array}$$

Furthermore, if θ has a C-approximating sequence and $S : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is a completely bounded and θ-equivariant operator, then

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_\theta G) & \xrightarrow{\ j_p\ } & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)) \\ \Big\downarrow {\scriptstyle (S \rtimes \mathrm{Id})} & & \Big\downarrow {\scriptstyle (S_\alpha)^{\mathcal{U}}_\alpha} \\ L_p(\mathcal{M} \rtimes_\theta G) & \xrightarrow{\ j_p\ } & \prod_{\alpha, \mathcal{U}} L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)), \end{array}$$

where

$$\big\| S_\alpha : L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)) \to L_p(\mathcal{M} \,\overline{\otimes}\, \mathcal{B}(L_2 G)) \big\|_{\mathrm{cb}} \leq C^{\frac{1}{p}} \big\| S : L_p(\mathcal{M}) \to L_p(\mathcal{M}) \big\|_{\mathrm{cb}}.$$

Proof. Let j_p^α be the maps defined in Lemma 4.2.1, we define the isometry j_p just by $j_p = (j_p^\alpha)^{\mathcal{U}}_\alpha$ for some principal ultrafilter $\mathcal{U}$. Such map is completely contractive since each j_p^α is. To prove that it is an isometry notice that, for any von Neumann algebra $\mathcal{N}$ we have

$$\prod_{\alpha, \mathcal{U}} L_p(\mathcal{N}) \subset \Big(\prod_{\alpha, \mathcal{U}} L_{p'}(\mathcal{N}) \Big)^*, \tag{4.2.3}$$

and the embedding is isometric. Indeed, such identity is a consequence, when $1 < p$, of the fact that the dual of the ultraproduct is larger than the ultraproduct of the duals, see [Pis03, pp. 59-63, (2.8.8)]. For $p = 1$, in addition, we have to use the injectivity of the ultraproduct construction, see [Pis03, pp. 59-63, (2.8.2)], and apply it to the inclusion $L_1(\mathcal{N}) \subset L_1(\mathcal{N})^{**}$. With identity (4.2.3) at hand, we have that

$$\begin{aligned} \|j_p\, x\|_{\prod_{\alpha,\mathcal{U}} L_p} &= \|j_p\, x\|_{(\prod_{\alpha,\mathcal{U}} L_{p'})^*} \\ &= \sup_{\|h\|_{p'} \leq 1} \big| \langle j_p\, x, h \rangle \big| \\ &\geq \sup_{\|y\|_{p'} \leq 1} \big| \langle j_p\, x, j_{p'}\, y \rangle \big| \end{aligned}$$

$$\begin{aligned} &= \sup_{\|y\|_{p'} \le 1} \lim_{\alpha,\mathcal{U}} \big|\big\langle j_p^\alpha\, x, j_{p'}^\alpha\, y \big\rangle\big| \\ &= \sup_{\|y\|_{p'} \le 1} \big|\big\langle x, y \big\rangle\big| \\ &= \|x\|_{L_p}. \end{aligned}$$

Therefore j_p is an isometry. The fact that it is a complete isometry follows by similar means.

The intertwining identity concerning M_m and T_m is trivial since all of the contractions j_p^α satisfy that

$$j_p^\alpha\,(\mathrm{Id} \rtimes T_m) = (\mathrm{Id} \otimes M_m)\, j_p^\alpha$$

and so does their ultraproduct j_p. The other relation is more delicate. The reason is that, if we want j_p^α to intertwine $S \rtimes \mathrm{Id}$ and $S \otimes \mathrm{Id}$, we need to impose S to be $\mathcal{M}_\alpha$-bimodular, where $\mathcal{M}_\alpha$ is the von Neumann algebra given by

$$\mathcal{M}_\alpha = \{\theta_g^{-1} x_\alpha(g)\}_{g \in G}'' \subset \mathcal{Z}(\mathcal{M}).$$

But such condition is too restrictive. To overcome such difficulty, we will assume that net $(x_\alpha)_\alpha$ comes from a C-approximating sequence. Then, for any α we can define the operator $Y_\alpha \in \mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G)$ given by

$$(Y_\alpha\, \xi)(g) = \begin{cases} \Big(P_{\alpha,g}^\perp + P_{\alpha,g}\, \dfrac{1}{\theta_g^{-1} x_\alpha(g)}\Big)\, \xi(g) & \text{when } g \in G\text{-supp}[x_\alpha] \\ \xi(g) & \text{otherwise,} \end{cases}$$

where $P_{\alpha,g} \in \mathcal{Z}(\mathcal{M})$ is the orthogonal projection onto the support of $x_\alpha(g)$. Clearly, we have that

$$\|Y_\alpha\|_{\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)} \le \max\big\{1, \operatorname*{ess\,sup}_x \mu\{g \in G : (x,g) \in F_\alpha\}^{\frac{1}{2}}\big\} < \infty$$

and since $Y_\alpha\, X_\alpha = X_\alpha\, Y_\alpha = \mathbf{1}_\mathcal{M} \otimes P_{G\text{-supp}[x_\alpha]}$ we obtain that

$$j_p^\alpha\,(S \rtimes \mathrm{Id}) \quad = \quad \underbrace{\mathrm{Ad}_{X_\alpha^{1/p}}\,(S \otimes \mathrm{Id})\, \mathrm{Ad}_{Y_\alpha^{1/p}}}_{S_\alpha}\, j_p^\alpha,$$

where Ad_S is the operator given by $\mathrm{Ad}_S(T) = S^*\, T\, S$. All that rest to do is to estimate the c.b. norm of S_α. We have

$$\|S_\alpha\|_{\mathrm{cb}} \le \big\|\mathrm{Ad}_{X_\alpha^{1/p}}\big\|_{\mathrm{cb}}\, \|S \otimes \mathrm{Id}\|_{\mathrm{cb}}\, \big\|\mathrm{Ad}_{Y_\alpha^{1/p}}\big\|_{\mathrm{cb}}. \tag{4.2.4}$$

Therefore, if $\lim_{\alpha,\mathcal{U}} \|\mathrm{Ad}_{X_\alpha^{1/p}}\|_{\mathrm{cb}}\, \|\mathrm{Ad}_{Y_\alpha^{1/p}}\|_{\mathrm{cb}} < \infty$, then the result follows. We have that

$$\begin{aligned} \big\|\mathrm{Ad}_{X_\alpha^{1/p}}\big\|_{\mathrm{cb}} \quad &\le \quad \|X_\alpha\|_{\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)}^{\frac{2}{p}} \quad \le \quad \Big(\operatorname*{ess\,inf}_x \mu\{g \in G : (x,g) \in F_\alpha\}\Big)^{-\frac{1}{p}} \\ \big\|\mathrm{Ad}_{Y_\alpha^{1/p}}\big\|_{\mathrm{cb}} \quad &\le \quad \|Y_\alpha\|_{\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)}^{\frac{2}{p}} \quad \le \quad \max\big\{1, \operatorname*{ess\,sup}_x \mu\{g \in G : (x,g) \in F_\alpha\}\big\}^{\frac{1}{p}}. \end{aligned}$$

Using property (4.1.3) in the definition of C-approximating sequence gives

$$\big\|S_\alpha : L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)) \to L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G))\big\|_{\mathrm{cb}} \le C^{\frac{1}{p}}\, \big\|S : L_p(\mathcal{M}) \to L_p(\mathcal{M})\big\|_{\mathrm{cb}}$$

and that concludes the proof. □

As a corollary we obtain that, for any amenable action, if M_m is a completely bounded Herz-Schur multiplier in $S_p(L_2G)$ then $\mathrm{Id} \rtimes T_m$ is c.b. in $L_p(\mathcal{M} \rtimes_\theta G)$. But [CdlS15, Theorem 4.2] asserts that if T_m is c.b in $L_p(\mathcal{L}G)$, so is M_m in $S_p(L_2G)$. Therefore, we get that if T_m is c.b. so is $\mathrm{Id} \rtimes T_m$. Similarly, if S is a θ-equivariant c.b. operator over $L_p(\mathcal{M})$ we have that $S \rtimes \mathrm{Id}$ is also c.b. The corollary bellow summarises both statements

Corollary 4.2.3. *Let $\theta : G \to \mathrm{Aut}(\mathcal{M})$ be an amenable action and G an unimodular group, for any $1 \leq p \leq \infty$, we have that*

$$\begin{aligned} & \left\|\mathrm{Id} \rtimes T_m : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \rtimes_\theta G)\right\|_{\mathrm{cb}} \\ \leq \; & \left\|M_m : L_p(\mathcal{B}(L_2G)) \to L_p(\mathcal{B}(L_2G))\right\|_{\mathrm{cb}} \\ \leq \; & \left\|T_m : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)\right\|_{\mathrm{cb}} \end{aligned} \tag{4.2.5}$$

Furthermore, if $S \in \mathcal{CB}(L_p(\mathcal{L}G))$ is θ-equivariant and θ as a C-approximating sequence, then

$$\begin{aligned} & \left\|S \rtimes \mathrm{Id} : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \rtimes_\theta G)\right\|_{\mathrm{cb}} \\ \leq \; & C^{\frac{1}{p}} \left\|S : L_p(\mathcal{M}) \to L_p(\mathcal{M})\right\|_{\mathrm{cb}}. \end{aligned} \tag{4.2.6}$$

4.3 Stability of maximal bounds

The goal of this section is to prove that the standard assumptions, defined at 3.1.1, are stable under certain cross-products. Let (H, ψ_H, X_H) and (G, ψ_G, X_G) be triples satisfying the standard assumption and let $\theta : G \to \mathrm{Aut}(H)$ be a μ_H-preserving action. Then, $K = H \rtimes_\theta G$ is again an unimodular group and it is trivial to check that its Haar measure μ_K can be identified with $\mu_H \otimes \mu_G$, see [Wol07, Proposition 3.3.10]. The action θ extends to a normal and τ_H-preserving action on $\mathcal{L}H$. Let $\phi : H \to \mathbb{C}$ be a function inducing a normal c.b. multiplier T_ϕ over $\mathcal{L}H$. T_ϕ is θ-equivariant, i.e: $T_\phi(\theta_g(x)) = \theta_g(T_\phi(x))$, iff ϕ is θ-invariant, i.e.: $\phi(\theta_g(h)) = \phi(h)$. Therefore, if $\phi_1 : H \to \mathbb{C}$ and $\phi_2 : G \to \mathbb{C}$ are functions of positive type, the function $\phi : K \to \mathbb{C}$ given by

$$\phi(h, g) = \phi_1(h)\, \phi_2(g)$$

is also of positive type when ϕ_1 is θ-invariant. Indeed, let $\{(h_i, g_i)\}_{i=1}^n \subset K$, then

$$\begin{aligned} \left[\phi\big((h_i, g_i)^{-1}\, (h_j, g_j)\big)\right]_{i,j} &= \left[\phi\big(\theta_{g_i^{-1}}(h_i^{-1}\, h_j), g_i^{-1}\, g_j\big)\right]_{i,j} \\ &= \left[\phi_1(h_i^{-1}\, h_j)\, \phi_2(g_i^{-1}\, g_j)\right]_{i,j} \quad \geq \quad 0. \end{aligned} \tag{4.3.1}$$

The positivity of the matrix in the last line follows from the fact that the Schur product respects positivity. Taking $\phi_1 = e^{-t\psi_H}$ and $\phi_2 = e^{-t\psi_G}$ and using Theorem 1.6.1 gives that $\psi : K \to \mathbb{R}_+$ given by $\psi(h, g) = \psi_H(h) + \psi_G(g)$ is a c.n. length when ψ_H is θ-invariant. The next logical step in order to extend the standard assumptions to crossed products is to find a way of defining operators $X_1 \rtimes \mathbf{1}, \mathbf{1} \rtimes X_2 \in \mathcal{L}K_+^\wedge$ given $X_1 \in \mathcal{L}H_+^\wedge$ and $X_2 \in \mathcal{L}G_+^\wedge$. Notice that if $\pi : \mathcal{N} \to \mathcal{R}$ is any normal $*$-homomorphism between von Neumann algebras, then π extends to a normal order-preserving map $\pi : \mathcal{N}_+^\wedge \to \mathcal{R}_+^\wedge$. Therefore, it makes sense to apply the $*$-automorphisms θ_g to X_H. We will say that X_H is θ invariant if $\theta_g(X_H) = X_H$ for every $g \in G$. Again, extending the normal inclusions $\iota_1 : \mathcal{M} \hookrightarrow \mathcal{M} \rtimes_\theta G$ and $\iota_2 : \mathcal{L}G \to \mathcal{M} \rtimes_\theta G$ to the extended positive cone gives operators

$$\begin{aligned} \iota_1(X_H^2) &:= X_H^2 \rtimes \mathbf{1} \in \mathcal{L}K_+^\wedge \\ \iota_2(X_G^2) &:= \mathbf{1} \rtimes X_G^2 \in \mathcal{L}K_+^\wedge. \end{aligned}$$

and we define the metric $X \in \mathcal{L}K_+^\wedge$ by the following equation

$$X^2 = X_H^2 \rtimes \mathbf{1} + \mathbf{1} \rtimes X_G^2.$$

Theorem 4.3.1. *Let (H, ψ_H, X_H) and (G, ψ_G, X_G) be triples satisfying the standard assumptions and $\theta : G \to \mathrm{Aut}(H)$ be a continuous, μ_H-preserving action. Assume that ψ_H and X_H are θ-invariant. Then, (K, ψ, X), defined as above, is also standard.*

In the theorem above it is trivial to prove the L_2-Gaussian bounds and doublingness of Φ_X. The key part are the completely bounded Hardy-Littlewood inequalities. In order to prove that, we are going to use an ℓ_∞-valued version of Theorem 4.2.2. Notice that we are not imposing amenability of the action θ because the standard assumptions force G to be amenable, see Remark 1.10.1, and hence any action is amenable. The stability result for maximal operators will be the following.

Theorem 4.3.2. *Let $\mathcal{M}$ be a hyperfinite von Neumann algebra, G a LCH unimodular group and $\theta : G \to \mathrm{Aut}(\mathcal{M})$ a $\tau_\mathcal{M}$-preserving action admitting a C-approximating sequence. Let (Ω_1, ν_1) and (Ω_2, ν_2) be measure spaces, $(T_\omega)_{\omega \in \Omega_1}$ be a family of completely positive Fourier multipliers and $(S_\omega)_{\omega \in \Omega_2}$ is a family of completely positive and θ-invariant operators satisfying that*

$$\begin{array}{rcll} A & = & \big\|(T_\omega) : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G; L_\infty(\Omega_1))\big\|_{\mathrm{cb}} & < \infty \\ B & = & \big\|(S_\omega) : L_p(\mathcal{M}) \to L_p(\mathcal{M}; L_\infty(\Omega_2))\big\|_{\mathrm{cb}} & < \infty. \end{array} \tag{4.3.2}$$

Then, we have that

$$\begin{array}{rl} & \big\|(S_\omega \rtimes T_\zeta)_{(\omega,\zeta)} : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \rtimes_\theta G; L_\infty(\Omega_1) \otimes_{\min} L_\infty(\Omega_2))\big\|_{\mathrm{cb}} \\ \leq & C^{\frac{1}{p}}\, A\, B. \end{array}$$

Observe that, in the abelian case with trivial action $\theta = \mathbf{1}$, Theorem 4.3.2 follows by routinely applying Fubini-type arguments. We obtain the following corollary.

Corollary 4.3.3. *Let $\mathcal{M}$, G, θ, $(T_\omega)_{\omega \in \Omega}$ and $(S_\omega)_{\omega \in \Omega}$ be like in the previous theorem for some fixed (Ω, ν). We have that*

$$\big\|(S_\omega \rtimes T_\omega)_\omega : L_p(\mathcal{M} \rtimes_\theta G) \to L_p(\mathcal{M} \rtimes_\theta G; L_\infty(\Omega))\big\|_{\mathrm{cb}} \leq C^{\frac{1}{p}}\, A\, B,$$

where A and B are defined like in (4.3.2).

Recall that, since each T_ω above is a Fourier multiplier, there is an essentially unique symbol m_ω such that $T_\omega = T_{m_\omega}$. In order to prove the theorems above we need some preliminary results. We will use the following characterization of boundedness for L_∞-valued Schur multipliers whose proof we omit.

Proposition 4.3.4. *Let $(T_{m_\omega})_{\omega \in \Omega} \subset \mathcal{CB}(L_1(\mathcal{L}G))$, we have that $(M_{m_\omega})_\omega : S_p(L_2 G) \to S_p[L_\infty(\Omega)]$ boundedly iff for every $a \in S_p^k$ and $(b^\omega)_\omega \in S_{p'}^k[L_1(\Omega)]$ we have that*

$$\Big| \int_\Omega \sum_{i,j}^{k} m_\omega(g_i^{-1} g_j)\, a_{ij}\, b_{ij}^\omega \, d\,\mu(\omega) \Big| \leq K \, \|a\|_{S_p^k} \, \|(b^\omega)_\omega\|_{S_{p'}^k[L_1]}. \tag{4.3.3}$$

Furthermore, the optimal K satisfies that

$$K = \big\|(M_{m_\omega})_\omega : S_p \to S_p[L_\infty(\Omega)]\big\|.$$

The analogous results for complete norms follows after taking $a_{i\,j} \in S_p^m$ and $b_{ij}^\omega \in S_p^m$ in (4.3.3)

The following theorem is just a vector-valued extension of Theorem 4.2.2.

Theorem 4.3.5. *Let $\mathcal{M}$ be a hyperfinite von Neumann algebra and G, θ be as above with θ amenable. For any $1 \leq p < \infty$ and any operator space E we have a complete isometry*

$$L_p(\mathcal{M} \rtimes_\theta G; E) \xrightarrow{\ j_p\ } \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); E).$$

Furthermore, when E is an operator system j_p is completely positive.

If $E = C(X_i)$ is any abelian C^*-algebra, where X_i are compact Hausdorff spaces, and $(T_{m_x})_{x \in X_2}$ is a family of Fourier multipliers in $L_p(\mathcal{L}G)$, then the following diagram commute

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_\theta G; C(X_1)) & \xrightarrow{\ j_p\ } & \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1)) \\ \Big\downarrow {\scriptstyle (\mathrm{Id} \rtimes T_{m_x})_{x \in X_2}} & & \Big\downarrow {\scriptstyle (\mathrm{Id} \otimes M_{m_x})_{x \in X_2}} \\ L_p(\mathcal{M} \rtimes_\theta G; C(X_1 \times X_2)) & \xrightarrow{\ j_p\ } & \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1 \times X_2)), \end{array}$$

where M_{m_x} is the Herz-Schur multiplier associated with the symbol m_x. Furthermore, if θ has a C-approximating sequence and $(S_x)_{x \in X_2}$ are θ-equivariant operators in $L_p(\mathcal{M})$, then

$$\begin{array}{ccc} L_p(\mathcal{M} \rtimes_\theta G; C(X_1)) & \xrightarrow{\ j_p\ } & \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1)) \\ \Big\downarrow {\scriptstyle (S_x \rtimes \mathrm{Id})_{x \in X_2}} & & \Big\downarrow {\scriptstyle (S_x^\alpha)_{x \in X_2}^{\alpha,\mathcal{U}}} \\ L_p(\mathcal{M} \rtimes_\theta G; C(X_1 \times X_2)) & \xrightarrow{\ j_p\ } & \prod_{\alpha,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1 \times X_2)), \end{array}$$

where $(S_x^\alpha)_{x \in X_2}$ satisfies that

$$\begin{aligned} & \big\| (S_x^\alpha)_{x \in X_2} : L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1)) \to L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2 G); C(X_1 \times X_2)) \big\|_{\mathrm{cb}} \\ \leq \quad & C^{\frac{1}{p}} \, \big\| (S_x)_{x \in X_2} : L_p(\mathcal{M}; C(X_1)) \to L_p(\mathcal{M}; C(X_1 \times X_2)) \big\|_{\mathrm{cb}}. \end{aligned}$$

Before going into the proof we would like to clarify why we choose $C(X)$-valued operators instead of $L_\infty(\Omega)$-valued, for some measure space Ω, if all we care about are maximal bounds. The reason is that, in order to pass from the strong maximal type arguments in Theorem 4.3.2 to the Corollary 4.3.3 we need to restrict the maximal operator $(S_\omega \rtimes T_\zeta)_{(\omega,\zeta)}$ to the diagonal $\omega = \zeta$. This requires that the diagonal restriction operator $m : L_\infty(\Omega) \otimes L_\infty(\Omega) \to L_\infty(\Omega)$, given by $m(f \otimes g) = f\, g$, to be completely bounded. That is not the case is we take $L_\infty(\Omega) \overline{\otimes} L_\infty(\Omega) = L_\infty(\Omega)$. Nevertheless it holds if we take $L_\infty(\Omega) \otimes_{\min} L_\infty(\Omega)$, which is not a von Neumann algebra.

In order to prove Theorem 4.3.5 we will need the following well-known lemma, whose proof we omit.

Lemma 4.3.6 ([Pis98]). *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be hyperfinite von Neumann algebras and E an operator space. If $\psi : L_p(\mathcal{M}_1) \to L_p(\mathcal{M}_2)$ is a completely bounded map, then $\psi \otimes \mathrm{Id}_E : L_p(\mathcal{M}_1; E) \to L_p(\mathcal{M}_2; E)$ is completely bounded. Furthermore, if E is an operator system, the map $\psi \mapsto \psi \otimes_E$ preserves complete positive maps.*

Remark 4.3.7. When $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ is hyperfinite and $p = 1$, every map ϕ satisfying that $\phi \otimes \mathrm{Id}_E$ is bounded for every E is actually completely bounded, the same follows for $p = \infty$ when ψ is normal. For general p, the maps ψ satisfying that $\|\psi \otimes \mathrm{Id}_E : L_p(\mathcal{M}; E) \to L_p(\mathcal{M}; E)\| < \infty$, when $E = \ell_\infty$, are called *regular maps* and were studied in [Pis95b]. Such maps are exactly those which can be expressed as linear combinations of completely positive ones. In the non-hyperfinite case the theorem above is false. Indeed, in [Haa85], Haagerup proved that all the completely bounded maps in $\mathcal{M}$ are linear combinations of completely positive maps iff $\mathcal{M}$ is hyperfinite.

Proof (of Theorem 4.3.5). Let $(x_\alpha)_\alpha$ be an approximating sequence for the amenable action θ. We can construct X_α as in the proof of Theorem 4.2.2. For each j_p^α by

$$j_p^\alpha = (\mathrm{Ad}_{X_\alpha^{1/p}}\, j) \otimes \mathrm{Id}_E$$

and by Lemma 4.3.6 such maps are complete contractions, i.e.

$$\left\| j_p^\alpha : L_p(\mathcal{M} \rtimes_\theta G; E) \to L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G); E) \right\|_{\mathrm{cb}} \leq 1.$$

They are also completely positive when E is an operator system. Let us denote temporarily such maps by $j_{p,E}^\alpha$. Now it is enough to prove that

$$\lim_{\alpha, \mathcal{U}} \left\langle (j_{p,\mathcal{B}(H)}^\alpha\, x), (j_{p',S_1(H)}^\alpha\, y) \right\rangle = \langle x, y \rangle, \tag{4.3.4}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $L_p(\mathcal{M} \rtimes_\theta G; S_1(H))$ and $L_{p'}(\mathcal{M} \rtimes_\theta G; \mathcal{B}(H))$. That case suffices since we can always embed E in a completely isometric way inside $\mathcal{B}(H)$. The proof of (4.3.4) follows like in the scalar case. The identity implies that $j_p = (j_p^\alpha)_\alpha^{\mathcal{U}}$ is isometric since we can use that

$$\begin{array}{lcll} \displaystyle\prod_{\alpha,\mathcal{U}} L_p(\mathcal{M}; \mathcal{B}(H)) & \subset & \left(\displaystyle\prod_{\alpha,\mathcal{U}} L_{p'}(\mathcal{M}; S_1(H)) \right)^* & \text{when } 1 < p \leq \infty \\ \displaystyle\prod_{\alpha,\mathcal{U}} L_1(\mathcal{M}; \mathcal{B}(H)) & \subset & \left(\displaystyle\prod_{\alpha,\mathcal{U}} L_\infty(\mathcal{M}; S_1(H)^{**}) \right)^* & \text{otherwise} \end{array}$$

and proceed like in the proof of Theorem 4.2.2. The commutation identities follow similarly. $\square$

Theorems 4.3.5 gives a way of transferring bounds from $\mathrm{Id} \otimes M$ where M is a $C(X)$-valued Schur multiplier to $\mathrm{Id} \rtimes T$, where T is its associated $C(K)$-valued Fourier multiplier. In order to bound the maximal operator given by Schur multipliers $(\mathrm{Id}_\mathcal{M} \otimes M_{m_\omega})_{\omega \in \Omega}$ we need the following transference result generalizing [CdlS15, Theorem 4.2] to the L_∞-valued case.

Theorem 4.3.8. *Let G be a LCH and unimodular group, Ω a measure space and $(T_{m_\omega})_{\omega \in \Omega} \subset \mathcal{CB}(L_1(\mathcal{L}G))$ a family of Fourier multipliers. If $(M_{m_\omega})_{\omega \in \Omega}$ is the associated family of Schur multipliers then, for every $1 \leq p \leq \infty$*

$$\begin{aligned} & \left\| (M_{m_\omega})_{\omega \in \Omega} : S_p(L_2G) \to S_p(L_2G; L_\infty(\Omega)) \right\|_{\mathrm{cb}} \\ \leq \;& \left\| (T_{m_\omega})_{\omega \in \Omega} : L_p(\mathcal{L}G), \, L_p(\mathcal{L}G; L_\infty(\Omega)) \right\|_{\mathrm{cb}}. \end{aligned}$$

Proof. Let μ be a probability measure over Ω such that $L_1(\Omega, \mu)^* = L_\infty(\Omega)$, by [CdlS15, Lemma 4.1] there is a dense subset of exponents $1 \leq p \leq \infty$ such that we can choose sequences x_n and y_n of norm one elements in $L_p(\mathcal{L}G)$ and $L_{p'}(\mathcal{L}G)$ such that

$$\lim_n \langle y_n, T_m x_n \rangle = m(e).$$

Let us define $z_n = y_n \otimes \chi_\Omega \in L_p(\mathcal{L}G; L_1(\Omega,\mu))$. Since the $L_1(\Omega; L_p(\mathcal{L}G))$-norm is larger then the $L_p(\mathcal{L}G; L_1(\Omega))$-norm we obtain that $\|z_n\|_{L_p(\mathcal{L}G;L_1)} \leq 1$. Furthermore, if $(T_{m_\omega})_{\omega\in\Omega}$ is a family of multiplier as in the hypothesis, then

$$\lim_n \langle z_n, T_{m_\omega} x_n\rangle = \int_\Omega m_\omega(e)\, d\,\mu(\omega), \tag{4.3.5}$$

where the parying $\langle\cdot,\cdot\rangle$ is the duality pairing between $L_p(\mathcal{L}G; L_\infty)$ and $L_{p'}(\mathcal{L}G; L_1)$. Proving

$$\begin{aligned} & \Big| \int_\Omega \sum_{i,j}^{k} m_\omega(g_i^{-1} g_j)\, a_{ij}\, b_{ij}^\omega \, d\,\mu(\omega) \Big| \\ \leq\ & \big\|(T_\omega)_\omega : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G; L_\infty(\Omega))\big\|_{\mathrm{cb}} \, \|a\|_{S_p^k} \, \|(b^\omega)_\omega\|_{S_p^k[L_1]} \end{aligned} \tag{4.3.6}$$

implies, by Proposition 4.3.3, that

$$\|(M_{m_\omega})_\omega\|_{\mathcal{B}(S_p,\, S_p(L_\infty))} \leq \big\|(T_{m_\omega})_\omega : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G; L_\infty(\Omega))\big\|_{\mathrm{cb}} \, \|a\|_{S_p^k} \|(b^\omega)_\omega\|_{S_p^k[L_1]}.$$

In order to obtain the same bound for the complete norms it is enough to take $a_{ij} \in S_p^m$ and repeat the calculation. Therefore to prove the claim it suffices to prove (4.3.6). Pick x_n and z_n like in (4.3.5) and consider $A_n \in S_p^k[L_p(\mathcal{L}G)]$ and $B_n^\omega \in S_{p'}^k[L_{p'}(\mathcal{L}G; L_1(\Omega))]$ given by

$$\begin{aligned} A_n &= u^*\,(a \otimes x_n)\, u \\ B_n^\omega &= u^*\,(b^\omega \otimes z_n)\, u \end{aligned}$$

where u is the unitary in $M_k \otimes \mathcal{L}G$ given by

$$u = \begin{pmatrix} \lambda_{g_1} & 0 & \cdots & 0 \\ 0 & \lambda_{g_2} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_{g_k} \end{pmatrix}$$

As a consequence $\|A_n\|_{S_p^k[L_p(\mathcal{L}G)]} = \|x_n\|_{L_p(\mathcal{L}G)}$ and $\|B_n\|_{S_{p'}^k[L_{p'}(\mathcal{L}G;L_1)]} \leq \|z_n\|_{L_{p'}(\mathcal{L}G;L_1)}$. Observe that $\lambda_{g_i}\, T_m(\lambda_{g_i}^* x \lambda_{g_j})\lambda_{g_j}^* = T_{m_{ij}}(x)$, where $m_{ij}(h) = m(g_i^{-1}\, h\, g_j)$, therefore

$$\begin{aligned} \int_\Omega \sum_{i,j}^{k} m_\omega(g_i^{-1} g_j) a_{ij} b_{ij}^\omega \, d\,\mu(\omega) &= \int_\Omega \sum_{i,j}^{k} a_{ij} b_{ij}^\omega \lim_n \langle z_n, T_{m_{ij}^\omega} x_n\rangle \, d\,\mu(\omega) \\ &= \lim_n \int_\Omega \sum_{i,j}^{k} a_{ij} b_{ij}^\omega \langle z_n, T_{m_{ij}^\omega} x_n\rangle \, d\,\mu(\omega) \\ &= \lim_n \langle (B_n^\omega)_\omega, ((\mathrm{Id} \otimes T_{m_\omega})A_n)_\omega\rangle \\ &\leq \big\|(T_{m_\omega})_\omega : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G; L_\infty)\big\|_{\mathrm{cb}} \, \|a\|_{S_p^k} \|(b^\omega)_\omega\|_{S_p^k[L_1]}. \end{aligned}$$

We have used the Dominated Convergence Theorem to exchange the limit and the integral in the second equation, which is justified since the multipliers m_ω are uniformly bounded. $\square$

We can pass to the proof of the strong maximal bounds. Since we are going to reduce the problem to that of tensor product it is convenient to recall the following modification of the result for tensor products, see [GPJP15, Lemma 2.8], whose proof we omit.

Theorem 4.3.9. *Let (M_i, τ_i), for $i \in \{1, 2\}$ be two hyperfinite von Neumann algebras with n.s.f. traces, (Ω_i, ν_i) two measure spaces and $(S_\omega)_{\omega\in\Omega_1}$, $(T_\omega)_{\omega\in\Omega_2}$ be families of completely positive operators satisfying that*

$$\begin{array}{rcll} A_1 & := & \left\|(T_\omega)_{\omega\in\Omega_1} : L_p(\mathcal{M}_1) \to L_p(\mathcal{M}_1; L_\infty(\Omega_1))\right\|_{\mathrm{cb}} & < \infty \\ A_2 & := & \left\|(S_\omega)_{\omega\in\Omega_2} : L_p(\mathcal{M}_2) \to L_p(\mathcal{M}_2; L_\infty(\Omega_2))\right\|_{\mathrm{cb}} & < \infty \end{array}.$$

Then, we have that

$$\left\|(R_{\omega,\zeta})_{(\omega,\zeta)} : L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2) \to L_p(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2; L_\infty(\Omega_1) \otimes_{\min} L_\infty(\Omega_2))\right\|_{\mathrm{cb}} \leq A_1\, A_2,$$

where

$$R_{\omega,\zeta} = \mathrm{Ad}_Y\,(T_\omega \otimes \mathrm{Id})\,\mathrm{Ad}_X\,(\mathrm{Id} \otimes S_\zeta)$$

and $X, Y \in \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ are self adjoint contractive operators.

Proof (of Theorem 4.3.2). Recall that for any measure space Ω, the algebra $L_\infty(\Omega)$ is isomorphic to $C(X)$ where X is certain compact Hausdorff and disconnected topological space. In order to apply Theorem 4.3.5 we need to express an element $\omega \mapsto T_\omega$ inside $L_\infty(\Omega; \mathcal{CB}(L_p(\mathcal{N})))$ as a $\mathcal{CB}(L_p(\mathcal{N}))$-valued function on $C(X)$. But, since $X \subset \mathrm{Ball}(L_\infty(\Omega)^*)$, we can safely evaluate $\phi \otimes \mathrm{Id}$, where $\phi \in X$, against $(T_\omega)_\omega$. By an application of Theorem 4.3.5 the diagram below commutes.

$$\begin{tikzcd}
L_p(\mathcal{M} \rtimes_\theta G) \arrow[rr, "(S_\omega \rtimes T_\zeta)_{(\omega,\zeta)}"] \arrow[dr, "(S_\omega \rtimes \mathrm{Id})_t"] \arrow[ddd, "j_p"] & & L_p(\mathcal{M} \rtimes_\theta G; L_\infty \otimes_{\min} L_\infty) \arrow[ddd, "j_p"] \\
& L_p(\mathcal{M} \rtimes_\theta G; L_\infty) \arrow[ur, "(\mathrm{Id} \rtimes T_\zeta)_\zeta"] \arrow[d, "j_p"] & \\
& \prod_{n,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G); L_\infty) \arrow[dr, "(\mathrm{Id} \otimes T_\zeta)_\zeta"] & \\
\prod_{n,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G)) \arrow[ur, "(S_\omega^\alpha)_\omega^{\alpha,\mathcal{U}}"] & & \prod_{n,\mathcal{U}} L_p(\mathcal{M} \overline{\otimes} \mathcal{B}(L_2G); L_\infty \otimes_{\min} L_\infty)
\end{tikzcd}$$

The j_p are the complete isometries of Theorem 4.3.5. The isometries j_p intertwine $(S_\omega \rtimes \mathrm{Id})$ with the ultraproduct with respect to $\mathcal{U}$ in α of the maps

$$S_\omega^\alpha = \mathrm{Ad}_{X_\alpha^{1/p}}\,(S_\omega \otimes \mathrm{Id})\,\mathrm{Ad}_{Y_\alpha^{1/p}}$$

and so $(S_\omega \rtimes \mathrm{Id})_{\omega\in\Omega_2}$ is completely bounded (resp. completely positive) if the ultraproduct of such maps is completely bounded (resp. completely positive). But, since each S_ω is c.p. and $\mathcal{M}$ is hyperfinite that follows by Lemma 4.3.6. Similarly, $(\mathrm{Id} \rtimes T_\zeta)_{\zeta\in\Omega_2}$ is completely bounded (resp.

completely positive) if $(\mathrm{Id} \otimes M_\zeta)_{\zeta \in \Omega_2}$ is c.b. (resp. c.p.), where M_ζ is the Schur multiplier sharing its symbol with T_ζ. By Theorem 4.3.8 $(\mathrm{Id} \otimes M_\zeta)_\zeta$ is completely bounded. Now, applying Theorem 4.3.9 gives that $((\mathrm{Id} \otimes M_\zeta)\, S^\alpha_\omega)_{(\omega,\zeta)}$ is completely bounded and that finishes the proof. □

The Corollary 4.3.3 follows from the Theorem above after applying Lemma 3.1.3.

Proof (of Corollary 4.3.3). Notice that, if $\mathcal{R}_1 = (S_\omega \rtimes T_\zeta)_{(\omega,\zeta)}$ and $\mathcal{R}_2 = (S_\omega \rtimes T_\omega)_\omega$, we have that:

$$\mathcal{R}_2 = (\mathrm{Id}_{\mathcal{M} \rtimes G} \otimes m)\, \mathcal{R}_1,$$

and applying Lemma 3.1.3 together with Theorem 4.3.2 gives the desired result. □

With that at hand we can pass to prove of the stability under crossed products of the standard assumptions.

Proof (of Theorem 4.3.1). To prove that X is doubling we just use that $X_H^2 \rtimes \mathbf{1}$ and $\mathbf{1} \rtimes X_G^2$ commute when X_H is θ invariant and therefore:

$$\chi_{[0,r^2)}(X) \leq \chi_{[0,r^2)}(X_H \rtimes \mathbf{1})\, \chi_{[0,r^2)}(\mathbf{1} \rtimes X_G).$$

Using that $\tau_K((x \rtimes \mathbf{1})\,(\mathbf{1} \rtimes y)) = \tau_H(x)\, \tau_G(y)$ gives that $\Phi_X(r) \leq \Phi_{X_H}(r)\Phi_{X_G}(r)$. Similarly it can be proved that $\Phi_{X_H}(r)\Phi_{X_G}(r) \leq \Phi_X(2\, r)$ and therefore X is doubling. The L_2GB property is proved similarly. For the CBHL maximal inequalities we just use that

$$\frac{\chi_{[0,r]}(X)}{\Phi_X(r)} \star u \lesssim_{(D_{\Phi_{X_H}}, D_{\Phi_{X_G}})} (\mathcal{R}^H_r \rtimes \mathcal{R}^G_r)(u),$$

where

$$\mathcal{R}^H_r(u) \quad = \quad \frac{\chi_{[0,r]}(X_H)}{\Phi_{X_H}(r)} \star u \quad \text{and} \quad \mathcal{R}^G_r(u) \quad = \quad \frac{\chi_{[0,r]}(X_G)}{\Phi_{X_G}(r)} \star u.$$

The maximal boundedness of $(\mathcal{R}^H_r \rtimes \mathcal{R}^G_r)_{r \geq 0}$ follows from Corollary 4.3.3. □

4.4 Foreword

There are several open questions that have been left open in the preceding sections, apart from our conjecture claiming that every amenable action has a C-approximating sequence. Our plan is to attack some of those problems in our forthcoming work [GP16a].

The first of such questions is whether the existence of an isometry j_p with properties analogous to those stated in Theorem 4.2.2 implies amenability for the action.

Conjecture 4.4.1. If Γ is a discrete group, $\theta : G \to \mathrm{Aut}(\mathcal{M})$ is a trace-preserving action and there is an complete isometry

$$L_p(\mathcal{M} \rtimes_\theta \Gamma) \xrightarrow{\ j_p\ } \prod_{\alpha, \mathcal{U}} L_p(\mathcal{B}(\ell_2) \,\overline{\otimes}\, \mathcal{M} \,\overline{\otimes}\, \mathcal{B}(\ell_2\Gamma)),$$

for large enough p, satisfying that

$$j_p\,(\mathrm{Id} \rtimes T_m) = (\mathrm{Id} \otimes M_m)^{\alpha,\mathcal{U}}\, j_p,$$

then, the action θ is amenable.

If such result were true it would be a natural generalization of [CdlS15, Theorem 2.1]. We restrict ourselves to discrete groups since the connection between amenability and approximation properties of $\mathcal{L}G$ is more transparent there. Observe also that if $\mathcal{M} = \mathbb{C}$ the result follows trivially from [CdlS15, Theorem 2.1].

Another potential application of our theorems is to transfer complete bounds from equivariant operators in $L_p(\Omega)$, for some G-space Ω, to complete bounds for operators in $L_p(\mathcal{L}G)$. This is a two step method. First, we obtain complete bounds for $S \rtimes \mathrm{Id} : L_p(\Omega \rtimes_\theta G) \to L_p(\Omega \rtimes_\theta G)$ assuming that θ has a C-approximating sequence. Then, we use an isometric embedding $J : L_p(\mathcal{L}G) \to L_p(\Omega \rtimes_\theta G)$ to transfer such bound back to $L_p(\mathcal{L}G)$. Let us explain a little bit more the second step. If $\beta : G \to L_\infty(\Omega)$ satisfies that

$$\begin{aligned} \beta_e &= \mathbf{1}_\Omega \\ \beta_{g\,h} &= \beta_g\, \theta_g(\beta_h), \end{aligned}$$

we will say it is a multiplicative 1-cocycle with respect to θ. It is trivial to verify that the map $J_\beta(\lambda_g) = \beta_g \rtimes \lambda_g$ is multiplicaive and so, with mild measurability assumptions, we can extend J_β to a normal $*$-homomorphism $J_\beta : \mathcal{L}G \to L_\infty(\Omega) \rtimes_\theta G$. Let us assume, for the sake of the argument, that Ω is a probability space. Then J_β is trace preserving and so it extends to an isometry

$$L_p(\mathcal{L}G) \xrightarrow{\ J_\beta\ } L_p(\Omega \rtimes_\theta G).$$

If $T_m^\beta \in \mathcal{CB}(L_p(\Omega))$ is a θ-equivariant map sending $\beta_g \mapsto m(g)\,\beta_g$, for some symbol $m \in L_\infty(G)$, we have that

$$\begin{tikzcd} L_p(\mathcal{L}G) \arrow[r, hook, "J_\beta"] \arrow[d, "T_m"] & L_p(\Omega \rtimes G) \arrow[d, "T_m^\beta \rtimes \mathrm{Id}"] \\ L_p(\mathcal{L}G) \arrow[r, hook, "J_\beta"] & L_p(\Omega \rtimes G). \end{tikzcd} \tag{4.4.1}$$

Therefore, the associated multiplier $T_m : L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)$ will be completely bounded. Choosing the functions β_g from the spectral resolvent of a θ-equivariant operator on Ω, think of a Laplace-Beltrami operator in an homogeneous manifold, it will be possible to express T_m^β as some spectral multiplier and transfer complete bounds in L_p from such multiplier to $L_p(\mathcal{L}G)$. The principal obstruction is that Ω being a probability space forces G to be amenable and when Ω is infinite the map J_β is not trace-preserving and so, to extend it to L_p requires to use an approximate identity to obtain asymptotic isometries

$$L_p(\mathcal{L}G) \xrightarrow{\ (J_\beta^\alpha)_\alpha^\mathcal{U}\ } \prod_{\alpha,\mathcal{U}} L_p(\Omega \rtimes_\theta G).$$

It is easy to see that such isometries exist when G is amenable and that they satisfy the intertwining identities extending (4.4.1).

Problem 4.4.2. Does an isometry

$$L_p(\mathcal{L}G) \xrightarrow{\ (J_\beta^\alpha)_\alpha^\mathcal{U}\ } \prod_{\alpha,\mathcal{U}} L_p(\Omega \rtimes_\theta G),$$

satisfying the intertwining identities (4.4.1), exist for nonamenable G?

The solution of the problem above will be extremely interesting either with a negative or a positive answer. It is also worth recalling that even if the solution turns out to be negative there are more techniques to transfer bounds from $L_p(\Omega \rtimes G)$. For example, in [JMP14a], they transfer bounds for $\mathrm{BMO}(\Omega \rtimes_\theta G)$, an L_∞-like space, and then they use interpolation to get the whole L_p scale.

We have chosen to cover just the case of G unimodular and θ trace-preserving. Nevertheless, the ideas here presented have natural extension to the situation in which G is not unimodular, $\mathcal{M}$ is not semifinite and the action is just $\varphi_\mathcal{M}$-preserving up to scalar, i.e: $\varphi_\mathcal{M} \circ \theta_g = D_g \, \varphi_\mathcal{M}$, for some homomorphism $D : G \to \mathbb{R}_+^*$. In such case there is a weight φ generalizing both the Plancherel weight over $\mathcal{L}G \subset \mathcal{M} \rtimes_\theta G$ and $\varphi_\mathcal{M}$ over $\mathcal{M} \subset \mathcal{M} \rtimes_\theta G$. Furthermore the modular operator of φ can be obtained from D, the modular function of G and the modular operator associated to $\varphi_\mathcal{M}$.

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Chapter 5

Further Lines

The main result of this Chapter is that quantum relations, a concept introduced recently by Nik Weaver, see [Wea12], are in bijective correspondence with weak-$*$ closed left ideals inside $\mathcal{M} \otimes_{eh} \mathcal{M}$, where $\otimes_{eh}$ is the extended Haagerup tensor product. At the end of this chapter we will indicate how this knowledge can be use to describe off-diagonal restrictions of operators.

5.1 Prerequisites

We will recall some definitions and facts on quantum relations and on the Haagerup tensor product in this section.

5.1.1 Weaver's Quantum Relations

In [Wea12, KW12] Kuperberg and Weaver introduced the concept of a quantum relation over a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(H)$. They defined a quantum relation to be a weak-$*$ closed operator bimodule over $\mathcal{M}'$, i.e.: a linear weak-$*$ closed subset $\mathcal{V} \subset \mathcal{B}(H)$ satisfying that $\mathcal{M}' \, \mathcal{V} \, \mathcal{M}' \subset \mathcal{V}$. It is easy to see that such notion doesn't depend on the representation $\mathcal{M} \subset \mathcal{B}(H)$.

In the case $\mathcal{M} = \ell_\infty(X) \subset \mathcal{B}(\ell_2 X)$ acting by multiplication operators we have that $\mathcal{M}' = \mathcal{M}$. Identifying $\mathcal{B}(\ell_2 X)$ with matrices indexed by $X \times X$, gives that $\mathcal{V} \subset \mathcal{B}(\ell_2 X)$ is a quantum relation whenever

$$[a_{x\,y}]_{x,y \in X} \in \mathcal{V} \implies [b_x \, a_{x\,y} c_y]_{x,y \in X} \in \mathcal{V}, \tag{5.1.1}$$

for every $(b_x)_{x \in X}$, $(c_x)_{x \in X}$. This in turns easily implies, see [Wea12, Proposition 1.3], that there is a unique subset $R \subset X \times X$ such that

$$\mathcal{V}_R = \{[a_{x\,y}]_{x,y} : (x,y) \notin R \implies a_{x\,y} = 0\}.$$

and reciprocally every such subset $R \subset X \times X$ have associated the operator bimodule of all matrices supported on R. When $\mathcal{M} = L_\infty(X) \subset \mathcal{B}(L_2 X)$ is abelian but not atomic we do not have a bijective correspondence between $\mathcal{M}$ bimodules and measurable subsets of $X \times X$. In that case the natural object to substitute the (discrete) relations $R \subset X \times X$ will be the, so called, measurable relations, i.e. weak-$*$ open subsets $\mathcal{R} \subset \mathcal{P}(\mathcal{M}) \times \mathcal{P}(\mathcal{M})$ satisfying that

$$\Big(\bigvee_\alpha P_\alpha, \bigvee_\beta Q_\beta\Big) \in \mathcal{R} \Longleftrightarrow \exists \alpha_0, \beta_0 \, (P_{\alpha_0}, Q_{\beta_0}) \in \mathcal{R}.$$

The measurable relation associated with a quantum relation $\mathcal{V} \subset \mathcal{B}(L_2(X))$ is given by

$$\mathcal{R}_{\mathcal{V}} = \{(P,Q) \in \mathcal{P}(\mathcal{M}) \times \mathcal{P}(\mathcal{M}) : P\,\mathcal{V}\,Q \neq \{0\}\}. \tag{5.1.2}$$

Notice that in the abelian discrete case we have that $\mathcal{R}$ is just the set of projections (χ_A, χ_B) such that there are $x \in A$ and $y \in B$ with $(x,y) \in R$. Reciprocally, given any measurable relation $\mathcal{R}$ we can associate a quantum relation over $\mathcal{M}$ given by

$$\mathcal{V}_{\mathcal{R}} = \{T \in \mathcal{B}(L_2X) : P\,T\,Q = 0, \,\forall\,(P,Q) \notin \mathcal{R}\}. \tag{5.1.3}$$

It is proved in [Wea12] that the map $\mathcal{R} \mapsto \mathcal{V}_{\mathcal{R}}$ is injective. Unfortunately it is not surjective in general. This has to do with the fact that all the operator bimodules $\mathcal{V}$ arising like in 5.1.3 are not just weak-$*$ closed but *operator reflexive*, see [Erd86, Lar82] and in particular closed in the weak operator topology, or WOT in short. The way to fix that is to observe that if $\mathcal{V} \subset \mathcal{B}(H)$ is any weak-$*$ closed linear subspace $\mathbf{1} \otimes \mathcal{V} \subset \mathcal{B}(\ell_2 \otimes_2 H)$ is operator reflexive. Since $\mathbf{1} \otimes \mathcal{V}$ is a $\mathbb{C}\mathbf{1} \otimes \mathcal{M}'$-bimodule and $(\mathbb{C}\mathbf{1} \otimes \mathcal{M}')' = \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$ we have that $\mathbf{1} \otimes \mathcal{V}$ is a quantum relation over the amplified algebra $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$. This suggests that the right definition for quantum relations as pairs of related projections is given by amplified projections in $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$. The next definition captures this intuition.

Definition 5.1.1 ([Wea12, Definition 2.24]). We will say that $\mathcal{R} \subset \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2)) \times \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2))$ is an *intrinsic quantum relation* iff

(i) $\mathcal{R}$ is weak-$*$ open.

(ii) $(0,0) \notin \mathcal{R}$.

(iii) If $(P_\alpha)_{\alpha \in A}$ and $(Q_\beta)_{\beta \in B}$ are sets of families of projections in $\mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell_2))$ then

$$\Big(\bigvee_{\alpha \in A} P_\alpha, \bigvee_{\beta \in B} Q_\beta \Big) \in \mathcal{R} \iff \exists \alpha_0 \in A, \beta_0 \in B \text{ such that } (P_{\alpha_0}, Q_{\beta_0}) \in \mathcal{R}.$$

(iv) For every $B \in \mathbf{1} \otimes \mathcal{B}(\ell_2)$ we have that

$$([BP], Q) \in \mathcal{R} \iff (P, [B^* Q]) \in \mathcal{R},$$

where $[A]$ represents the left (or final) projection of the operator A.

Quantum relations over $\mathcal{M} \subset \mathcal{B}(H)$ and intrinsic quantum relations (or i.q.r.) over $\mathcal{M}$ are in bijective correspondence and the adaptations of the maps 5.1.2 and 5.1.3 are inverse of each other. Indeed, this correspondence works for every von Neumann algebra $\mathcal{M} \subset \mathcal{B}(H)$ not necessarily abelian or discrete, see [Wea12, Theorem 2.32].

Let X be a discrete measure space with the counting measure and let us identify $\mathcal{B}(\ell_2 X)$ with matrices indexed by X. Given a matrix $m = [m_{x\,y}]_{x\,y \in X}$ we define the *Schur multiplier* of symbol m as the operator S_m given by

$$S_m([a_{x\,y}]) = [m_{x\,y}\, a_{x\,y}].$$

Whenever S_m is completely bounded we will say that S_m is a c.b. Schur multiplier. We are going to denote by $\mathfrak{M}(X) \subset \mathcal{CB}(\mathcal{B}(\ell_2 X))$ the set of all c.b. Schur multipliers and by $\mathfrak{M}^\sigma(X)$ the space of all c.b. and normal ones (i.e. weak-$*$ continuous for $S_1(\ell_2 X)^* = \mathcal{B}(\ell_2 X)$). Assume that X is a finite set, let $R \subset X \times X$ be a relation and $\mathcal{V} \subset \mathcal{B}(\ell_2 X)$ be its associated quantum relation. We have that the ideal $J \subset \mathfrak{M}^\sigma(X) = \mathfrak{M}(X)$ given by

$$J = \{S \in \mathfrak{M}(X) : S|_{\mathcal{V}} = 0\} \tag{5.1.4}$$

contains just the Schur multipliers S_m whose symbol m satisfies that $m_{x\,y} = 0$ if $(x, y) \in R$. The reciprocal is also true and we have the following.

Proposition 5.1.1. *Let X be a finite set and $\ell_\infty(X) \subset \mathcal{B}(\ell_2 X)$ and $\mathcal{V} \subset \mathcal{B}(\ell_2 X)$ be as above. Then if J an ideal in $\mathfrak{M}^\sigma(X)$ we have that*

$$\begin{aligned} \mathcal{V}_J &= \{T \in \mathcal{B}(\ell_2^n) : S(T) = 0, \forall S \in J\} \\ J_\mathcal{V} &= \{S \in \mathfrak{M}^\sigma(X) : S|_\mathcal{V} = 0\} \end{aligned}$$

are bijections between the sets of quantum relations and the set of ideals of Schur multipliers. Furthermore, the maps $\mathcal{V} \mapsto J_\mathcal{V}$ and $J \mapsto \mathcal{V}_J$ are inverse of each other.

Such result was generalized to general, not necessarily abelian, finite dimensional von Neumann algebras $\mathcal{M} \subset \mathcal{B}(H)$ by Weaver [Wea12]. For that end recall that $\mathfrak{M}^\sigma(X)$ is actually equal to the algebra of all completely bounded normal operators $S : \mathcal{B}(\ell_2 X) \to \mathcal{B}(\ell_2 X)$ that are $\ell_\infty(X)$-bimodular. We are going to denote the the algebras of $\mathcal{M}'$-bimodular c.b. normal operators on $\mathcal{B}(H)$ by $\mathcal{CB}^\sigma_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$. It is trivial to see that in the case of finite dimensional $\mathcal{M}$ we have a bounded, quasi-isometric and multiplicative map $\Phi : \mathcal{M} \otimes_{\min} \mathcal{M}_{\mathrm{op}} \to \mathcal{CB}^\sigma_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$ given by extension of

$$\Phi(x \otimes y) = (T \mapsto xTy). \tag{5.1.5}$$

To see that, let $n = \dim \mathcal{M}$, so that, $\mathcal{B}(H)$ is quasi-isometric to $\ell_2^n \otimes_2 \ell_2^n$ and $\mathcal{CB}^\sigma(\mathcal{B}(H))$ is quasi isometric to $\mathcal{B}(\ell_2 \otimes_2 \ell_2)$. If $x, y \in \mathcal{M}'$, we denote by $T_{x\,y}$ the operator given by $S \mapsto xSy$. It is clear that $\phi \in \mathcal{B}(\ell_2 \otimes_2 \ell_2)$ is $\mathcal{M}'$-bimodular iff it belongs to the commutant of $\{T_{x\,y}\}_{x,y\in\mathcal{M}'}$ but such algebra is isomorphic to $\mathcal{M} \otimes \mathcal{M}_{\mathrm{op}}$ as we claimed. If $\mathcal{V} \subset \mathcal{B}(H)$ is a quantum relation over $\mathcal{M}$ we have that $J_\mathcal{V} = \{s \in \mathcal{M} \otimes \mathcal{M}_{\mathrm{op}} : \Phi_s|_\mathcal{V} = 0\}$ is a left ideal and therefore is of the form $J_\mathcal{V} = (\mathcal{M} \otimes \mathcal{M}_{\mathrm{op}})\, p_\mathcal{V}$ for some $p_\mathcal{V} \in \mathcal{P}(\mathcal{M} \otimes \mathcal{M}_{\mathrm{op}})$. Furthermore, we have the following.

Proposition 5.1.2 ([Wea12, Proposition 2.23]). *If $\mathcal{M}$ is finite dimensional the correspondence $\mathcal{V} \mapsto p_\mathcal{V}^\perp$ defined as above is an order-preserving bijection between quantum relations over $\mathcal{M}$ and projections in $\mathcal{M} \otimes \mathcal{M}_{\mathrm{op}}$.*

In the case of infinite dimensional von Neumann algebras $\mathcal{M}$ the result above fails and not every quantum relation can be associated with a projection in $\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}$. The reason for that is that although the map $\Phi : \mathcal{M} \otimes \mathcal{M}_{\mathrm{op}} \to \mathcal{CB}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$ is bounded and multiplicative for every finite dimensional algebra $\mathcal{M}$ it is far from isometric. Indeed its norm explodes with $n = \dim(\mathcal{M})$. The problem can be solved by changing the tensor norm from the spatial tensor norm to the Haagerup tensor norm of the two von Neumann algebras. With that tool at hand we will be able to prove a generalization of 5.1.1 for general algebras in the next section.

5.1.2 Module Maps and The Haagerup Tensor Product

Let E, F be two operator spaces. We define the bilinear form $\odot : M_n[E] \times M_n[F] \to M_n[E \otimes_{\mathrm{alg}} F]$ by

$$[x_{i\,j}] \odot [y_{i\,j}] = \Big[\sum_{k=1}^n x_{i\,k} \otimes y_{k\,j}\Big]_{i,j}.$$

Of course such definition makes perfect sense with matrices of different sizes $\odot : M_{n,m}[E] \times M_{m,l}[F] \to M_{n,l}[E \otimes_{\mathrm{alg}} F]$ just by embedding all matrices inside $M_{\max\{n,m,l\}}$ and restricting. The *Haagerup tensor norm* for $z \in E \otimes_{\mathrm{alg}} F$ is defined to be

$$\|z\|_h \;=\; \inf\{\|u\|_{M_{1,n}(E)}\, \|u\|_{M_{1,n}(E)} : z = u \odot v\}$$

$$= \quad \inf\left\{\Big\|\sum_{k=1}^{n} x_k x_k^*\Big\|^{\frac{1}{2}}\Big\|\sum_{k=1}^{n} y_k^* y_k\Big\|^{\frac{1}{2}} : z = \sum_{k=1}^{n} x_k \otimes y_k\right\}$$

The Haagerup tensor product $E \otimes_h F$ is defined as the completion under that norm. Similarly $E \otimes_h F$ can be given an o.s.s by defining:

$$\|x\|_{M_n[E\otimes_h F]} = \inf\{\|u\|_{M_{n,k}(E)}\,\|u\|_{M_{k,n}(E)} : z = u \odot v\}.$$

In the case of two dual operator spaces E^* and F^* the *weak-$*$ Haagerup tensor product*, introduced in [BS92b] by Blecher and Smith, is given by

$$E^* \otimes_{\mathrm{w}^*h} F^* = (E \otimes_h F)^*.$$

Since the Haagerup tensor norm is self dual, see [ER91], we have that $E^* \otimes_h F^*$ embeds inside $E^* \otimes_{\mathrm{w}^*h} F^*$ isometrically and is weak-$*$ dense. This tensor product is a complemented subspace of the *normal Haagerup tensor product* $E \otimes_{\sigma h} F$ introduced by Effros and Kishimoto [EK87] and which satisfies that

$$(E \otimes_h F)^{**} = (E^{**} \otimes_{\sigma h} F^{**}).$$

In [ER03] Effros and Ruan introduced the *extended Haagerup tensor product* generalizing the weak-$*$ Haagerup tensor to (potentially) non-dual operator spaces. Indeed if $x = [x_{i\,j}]_{i\,j}$ is a matrix whose entries are, possibly infinite, sums of simple tensors, we say that $x \in M_m(E \otimes_{eh} F)$ iff

$$\|x\|_{M_m(E\otimes_{eh} F)} = \inf\{\|u\|_{M_{m,I}(E)}\,\|v\|_{M_{m,I}(E)} : x = u \odot v\}$$

for every possible index set I. It can be seen that it is enough to take I to be the smallest cardinality of a dense set in H with $E, F \subset \mathcal{B}(H)$. Particularly when E and F are separable von Neumann algebras we can take I numerable. In the case of E^*, F^* being dual operator spaces, we have that

$$\begin{array}{rcl} E^* \otimes_{\mathrm{w}^*h} F^* & = & E^* \otimes_{eh} F^* \\ E^* \otimes_{\sigma h} F^* & = & (E \otimes_{eh} F)^*. \end{array}$$

The coarser topology in $E^* \otimes_{eh} F^*$ making the pairing with every element in $\mathcal{M}_* \otimes_{eh} \mathcal{M}_*$ continuous is strictly finer than the weak-$*$ topology given by the predual $\mathcal{M}_* \otimes_h \mathcal{M}_*$. Since $E^* \otimes_{eh} F^* \subset E^* \otimes_{\sigma h} F^*$ is $\sigma(E^* \otimes_{eh} F^*)$-closed, $E^* \otimes_{eh} F^*$, with the $\sigma(E^* \otimes_{eh} F^*)$ topology, is a dual space. Its predual is obtained by a quotient of $E^* \otimes_{eh} F^*$

When $E = \mathcal{N}$, $F = \mathcal{M}$ are von Neumann algebras $\mathcal{N} \otimes_{eh} \mathcal{M}$ is a weak-$*$ Banach algebra with a jointly completely bounded multiplication, see [ER00, pp. 126], given by extension of

$$(x \otimes y)\,(z \otimes t) = x\,z \otimes t\,y.$$

When $\mathcal{M} = \mathcal{N}$ there is also a natural multiplicative involution $(x \otimes y)^\dagger = y^* \otimes x^*$.

Recall that the space of completely bounded $\mathcal{CB}(E, F)$ has a natural o.s.s. given by the identification $M_n(\mathcal{CB}(E, F)) = \mathcal{CB}(E, M_n(F))$. If E^* and F^* are dual operator spaces we define $\mathcal{CB}^\sigma(E^*, F^*) \subset \mathcal{CB}(E^*, F^*)$ to be susbspace of all weak-$*$ continuous operators. We have a natural identification $\mathcal{CB}^\sigma(E^*, F^*) = \mathcal{CB}(F, E)$. When $E, F \subset \mathcal{B}(H)$ are bimodules over a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(H)$ we will denote by $\mathcal{CB}_{\mathcal{M}\,\mathcal{M}}(E, F)$ and $\mathcal{CB}^\sigma_{\mathcal{M}\,\mathcal{M}}(E, F)$ the subspaces of completely bounded and bimodular operators. Such subspaces are easily seen to be norm closed. We will treat mainly the case when $E = F = \mathcal{B}(H)$. We have, using that $\mathcal{K}(H)^{**} = \mathcal{B}(H)$ and that $\mathcal{CB}(E, F^*) = \mathcal{CB}^\sigma(E^{**}, F^*)$, see [BM04, (1.28)], that

$$\mathcal{CB}^\sigma(\mathcal{B}(H)) = \mathcal{CB}(\mathcal{K}(H), \mathcal{B}(H)). \tag{5.1.6}$$

The identification is given by restriction to $\mathcal{K}(H) \subset \mathcal{B}(H)$ and by passage to the second dual. The identity 5.1.6 allow us to give a predual for $\mathcal{CB}^{\sigma}(\mathcal{B}(H))$ by

$$\begin{aligned} \mathcal{CB}(\mathcal{K}(H), \mathcal{B}(H)) &= \mathcal{CB}(\mathcal{K}(H), \mathbb{C}) \otimes_{\mathcal{F}} \mathcal{B}(H) \quad \text{(by [Pis03, Th. 4.1])} \\ &= (\mathcal{K}(H) \widehat{\otimes} S_1(H))^*, \end{aligned} \tag{5.1.7}$$

where $\otimes_{\mathcal{F}}$ is the Fubini tensor product, see [EKR93], [ER03] or [ER00] which is isomorphic to the dual of the (operator space) projective tensor product $\widehat{\otimes}$, see [ER00, Chap. 7]. Similarly the predual of $\mathcal{CB}(\mathcal{B}(H))$ is given by $\mathcal{B}(H) \widehat{\otimes} S_1(H)$. In both cases the pairing is given by linear extension of $\langle T \otimes \xi, \Psi \rangle = \langle \xi, \Psi(T) \rangle$, for $\Psi \in \mathcal{CB}(\mathcal{B}(H))$. A subtle point is that the coarser topology in $\mathcal{CB}^{\sigma}(\mathcal{B}(H))$ making the paring with all the elements in $\mathcal{B}(H) \widehat{\otimes} S_1(H)$ continuous is, in general, strictly finer than the weak-$*$ topology given by the predual $\mathcal{K}(H) \widehat{\otimes} S_1(H)$. To see that, notice that the following inclusion holds

$$\mathcal{K}(H) \widehat{\otimes} S_1(H) \subset \mathcal{B}(H) \widehat{\otimes} S_1(H).$$

Indeed, the inclusion above is just a consequence of the fact that $\mathcal{K}(H) \subset \mathcal{B}(H)$ and the injectivity of the functor $E \mapsto \mathcal{M}_* \widehat{\otimes} E$, where $\mathcal{M}_*$ is the predual of any *hyperfinite* von Neumann algebra, see [Pis98]. Since $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$-closed sets are $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$-closed we have that $\mathcal{CB}^{\sigma}(\mathcal{B}(H)) \subset \mathcal{CB}(\mathcal{B}(H))$ is $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$-closed and so the $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ topology induces another predual for $\mathcal{CB}^{\sigma}(\mathcal{B}(H))$. Clearly, the topology of pointwise weak-$*$ convergence in $\mathcal{CB}^{\sigma}(\mathcal{B}(H))$ is coarser than the $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ topology. Analogously, the topology of pointwise (in $\mathcal{K}(H)$) weak-$*$ topology is coarser that the $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$ topology. In both cases the topologies coincide over bounded sets.

The subspace of bimodular operators $\mathcal{CB}^{\sigma}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H))$ is closed in both the $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$ and the $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ topologies. Indeed, it is closed in the $\mathcal{K}(H)$-pointwise weak-$*$ topology which is coarser than both. As a consequence, using the Hanhn-Banach Theorem, we get that $\mathcal{CB}^{\sigma}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H))$ inherits two natural preduals topologies

$$\begin{aligned} \mathcal{CB}^{\sigma}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H)) &= (\mathcal{B}(H) \widehat{\otimes} S_1(H)/K_2)^*, \\ \mathcal{CB}^{\sigma}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H)) &= (\mathcal{K}(H) \widehat{\otimes} S_1(H)/K_1)^*, \end{aligned}$$

where K_1, K_2 are the corresponding preannihilators. Similarly $\mathcal{CB}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H))$ is also a dual space with the $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ topology. The spaces $\mathcal{CB}_{\mathcal{M}\,\mathcal{M}}(\mathcal{K}(H))$, $\mathcal{CB}^{\sigma}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H))$ and $\mathcal{CB}_{\mathcal{M}\,\mathcal{M}}(\mathcal{B}(H))$ are Banach algebras with the composition operation. They have a natural multiplicative involution given by $\Psi_1^{\dagger}(T) = \Psi_1(T^*)^*$ and satisfying that $(\Psi_1 \Psi_2)^{\dagger} = \Psi_1^{\dagger} \Psi_2^{\dagger}$.

Example 5.1.3. Recall that in the case of $\mathcal{M} = \ell_\infty(X) \subset \mathcal{B}(\ell_2 X)$ we have that

$$\begin{aligned} \mathcal{CB}^{\sigma}_{\ell_\infty(X)\,\ell_\infty(X)}(\mathcal{B}(\ell_2 X)) &= \mathfrak{M}^{\sigma}(X), \\ \mathcal{CB}_{\ell_\infty(X)\,\ell_\infty(X)}(\mathcal{B}(\ell_2 X)) &= \mathfrak{M}(X). \end{aligned}$$

For non-discrete measure spaces (X, μ) we have that $\mathcal{CB}^{\sigma}_{L_\infty(X)\,L_\infty(X)}(\mathcal{B}(L_2 X))$ corresponds to the algebra of *measurable Schur multipliers*, see [Spr04].

Now we are in position of stating the isomorphism between Haagerup tensors and bimodular operators.

Theorem 5.1.4. *Let $\mathcal{M} \subset \mathcal{B}(H)$ be a von Neumann algebra. The map Φ defined by $x \otimes y \mapsto \Phi_{x \otimes y}$, where*

$$\Phi_{x \otimes y}(T) = x\,T\,y,$$

extends to a surjective complete isometry and a $\dagger$-preserving homomorphism between the following spaces

(i) $\Phi : \mathcal{M} \otimes_h \mathcal{M} \to \mathcal{CB}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{K}(H))$.

(ii) $\Phi : \mathcal{M} \otimes_{eh} \mathcal{M} \to \mathcal{CB}^{\sigma}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$.

(iii) $\Phi : \mathcal{M} \otimes_{\sigma h} \mathcal{M} \to \mathcal{CB}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$.

Furthermore, the map in (iii) *is* $\sigma(\mathcal{M}_* \otimes_{eh} \mathcal{M}_*)$ *to* $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ *continuous and the map in* (ii) *is both* $\sigma(\mathcal{M}_* \otimes_h \mathcal{M}_*)$ *to* $\sigma(\mathcal{K}(H) \widehat{\otimes} S_1(H))$ *continuous and* $\sigma(\mathcal{M}_* \otimes_{eh} \mathcal{M}_*)$ *to* $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ *continuous.*

The result above is well known to the experts, although their pieces are scattered throughout the literature. We will just give a brief sketch with references. Recall too that the first appearance of such result is credited to be in an unpublished note of Haagerup [Haa86].

Proof. Let us concentrate on (ii), which will be the most important for our applications. The fact that Φ is a complete contraction amounts to a trivial calculation. Indeed, if $s = \sum_j x_j \otimes y_j$ we may define, for every $1 \leq n$, the matrices

$$x = \sum_{i=0}^{n} \sum_j e_{i\,j} \otimes x_j, \quad y = \sum_{j=0}^{n} \sum_i e_{i\,j} \otimes x_i$$

inside $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$, where $\{e_{i\,j}\}$ is a system of matrix units. Then $(\mathrm{Id}_{M_n} \otimes \Phi_s)(T)$ satisfies that

$$(\mathrm{Id}_{M_n} \otimes \Phi_s)(T) = P_n\, x\, (\mathbf{1} \otimes T)\, y\, P_n,$$

where P_n is the orthogonal projection on the span of $\{e_j\}_{j \leq n}$. Clearly

$$\|\mathrm{Id}_{M_n} \otimes \Phi_s\| \leq \|x^* x\|_{\mathcal{M}}^{\frac{1}{2}}\, \|y\, y^*\|_{\mathcal{M}}^{\frac{1}{2}}$$

and Φ is an $\mathcal{M}'$-bimodular operator. Taking the supremum over $n \geq 1$ and the infimum over all representations of s gives that $\|\Phi_s\|_{\mathrm{cb}} \leq \|s\|_{\mathcal{M} \otimes_{eh} \mathcal{M}}$. To see that it is surjective notice that if $\Psi \in \mathcal{CB}^{\sigma}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H)) = \mathcal{CB}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{K}(H), \mathcal{B}(H))$ by Wittstock's factorization theorem for c.b. maps, see [Pau86], we have that there is a large enough ℓ_2 (we can take the dimension of ℓ_2 to be equal to that of H for infinite dimensional spaces), a representation $\pi : \mathcal{K}(H) \to \mathcal{B}(\ell_2 \otimes_2 H)$ and two elements $x \in \mathcal{B}(\ell_2 \otimes_2 H, H)$, $y \in \mathcal{B}(H, \ell_2 \otimes_2 H)$ such that $\Psi(x) = x\, \pi(x)\, y$ and $\|\Psi\|_{\mathrm{cb}} = \|x\|\, \|y\|$ but we can identify x and y with a row and a column respectively inside $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H)$ and we have that $\Psi = \Phi_s$, where $s = x \odot y \in \mathcal{B}(H) \otimes_{eh} \mathcal{B}(H)$. It only rest to prove that if Ψ is $\mathcal{M}'$-bimodular we can pick $x, y \in \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$, which is the main result in [Smi91, Theorem 3.1]. The rest of the points are similarly proved, see also [BS92b] for (iii). $\square$

As a consequence of the preceding theorem we are going to identify at times $\mathcal{M} \otimes_{eh} \mathcal{M}$ and its weak-$*$ topology with $\mathcal{CB}^{\sigma}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))$ and $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$. The following lemma describe the weak-$*$ continuous functionals on $\mathcal{M} \otimes_{eh} \mathcal{M}$ for its different preduals.

Lemma 5.1.5. *Let* $\phi \in (\mathcal{M} \otimes_{eh} \mathcal{M})^*$*, then*

(i) ϕ *is* $\sigma(\mathcal{M}_* \otimes_{eh} \mathcal{M}_*)$*-continuous iff*

$$\langle \phi, s \rangle = \big\langle C, (\mathrm{Id} \otimes \Phi_s)(B) \big\rangle,$$

where $C \in S_1(\ell_2 \otimes_2 H)$ *and* $B \in \mathcal{B}(\ell_2 \otimes_2 H)$

(ii) *ϕ is $\sigma(\mathcal{M}_* \otimes_h M_*)$-continuous iff*

$$\langle \phi, s\rangle = \big\langle C, (\mathrm{Id} \otimes \Phi_s)(B)\big\rangle,$$

where $C \in S_1(\ell_2 \otimes_2 H)$ and $B \in \mathcal{K}(\ell_2 \otimes_2 H)$

Furthermore, ϕ is pointwise weak-$*$ continuous, iff B in (i) *can be taken in $\mathcal{B}(\ell_2^n \otimes_2 H)$. Similarly, ϕ is $\mathcal{K}(H)$-pointwise weak-$*$ continuous iff we can take $B \in \mathcal{K}(\ell_2^n \otimes_2 H)$.*

Proof. We will prove (i) first. Since, by Theorem 5.1.4, the predual for the $\sigma(\mathcal{M}_* \otimes_h \mathcal{M}_*)$ topology is given by $(\mathcal{M} \otimes_{eh} \mathcal{M})_* = (\mathcal{K}(H) \widehat{\otimes} S_1(H))/F$, where F is the preannihilator of the $\mathcal{M}'$-bimodular maps, ϕ can be lifted to an element (that we will denote also by ϕ) in $\mathcal{K}(H) \widehat{\otimes} S_1(H)$ inducing the same functional. By definition of the o.s. projective tensor product we have that there are, possibly infinite, index sets I_1, I_2 and elements $A \in \mathcal{K}_{I_1} \otimes_{\min} S_1(H)$, $B \in \mathcal{K}_{I_2} \otimes_{\min} \mathcal{K}(H)$ and $\alpha, \beta \in S_2(\ell_2^{I_1}, \ell_2^{I_2})$, where $\mathcal{K}_{I_i} = \mathcal{K}(\ell_2^{I_i})$, such that

$$\phi = \sum_{i,j \in I_1\, p,q \in I_2} \alpha_{ip} (B_{ij} \otimes A_{pq}) \beta_{jq}.$$

The action on $s \in \mathcal{M} \otimes_{eh} \mathcal{M}$ is given by

$$\begin{aligned} \langle \phi, s\rangle &= \sum_{i,j \in I_1\, p,q \in I_2} \alpha_{ip} \, \langle A_{ij}, \Phi_s(B_{pq})\rangle \, \beta_{jq} \\ &= \sum_{p,q \in I_2} \Big\langle \sum_{i,j \in I_1} \bar{\alpha}_{ip} \, A_{ij} \, \beta_{jq}, \Phi_s(B_{pq})\Big\rangle \\ &= \big\langle (\alpha^* \otimes \mathbf{1}) \, A \, (\beta \otimes \mathbf{1}), (\mathrm{Id}_{\mathcal{K}_{I_2}} \otimes \Phi_s)(B)\big\rangle. \end{aligned}$$

Note that, by [Pis98, Theorem 1.5], $C = (\alpha^* \otimes \mathbf{1}) \, A \, (\beta \otimes \mathbf{1}) \in S_1(\ell_2^{I_2})[S_1(H)] \simeq S_1(\ell_2^{I_2} \otimes_2 H)$. We have thus that every weak-$*$ continuous functional ϕ can be expressed as

$$\langle \phi, s\rangle = \big\langle C, (\mathrm{Id}_{\mathcal{K}} \otimes \Phi_s)(B)\big\rangle,$$

concluding the proof of (i). The same techniques yield (ii).

The other claims in the statement follows by a repetition of the ideas used to prove that SOT-continuous and WOT-continuous functionals coincide over $\mathcal{B}(H)$. Indeed, assume ϕ is pointwise weak-$*$ continuous. Then, there are finite collection $T_1, ..., T_m \in \mathcal{B}(H)$ and $\xi_1, \xi_2, ..., \xi_m \in S_1(H)$ such that $|\phi(\Psi)| < 1$ whenever $|\langle \xi_i, \Psi(T_i)\rangle| < \epsilon$ for $i \in \{1, 2, ..., m\}$. In particular, taking $\Psi' = \Psi / \max\{|\langle \xi_i, \Psi(T_i)\rangle|\}$ gives

$$|\phi(\Psi)| \leq \epsilon^{-1} \max\{|\langle \xi_i, \Psi(T_i)\rangle|\} \leq \epsilon^{-1} \sum_{i=1}^{m} |\langle \xi_i, \Psi(T_i)\rangle|.$$

As a consequence, if $\Psi(T_i) = 0$, for $i \in \{1, 2, ..., m\}$, we have $\phi(\Psi) = 0$ and so ϕ factors through a finite dimensional space. Therefore, ϕ can be expressed as a finite combination of simple tensors. □

5.2 The Correspondence Between Ideals and Modules

In this section we are going to prove the correspondence between left ideals in $\mathcal{M} \otimes_{eh} \mathcal{M}$ and quantum relations over $\mathcal{M}$. We are going to start recalling two easy lemmas that will be thoroughly used in this section. The first asserts that the bilinear form $\odot$ can be extended from $M_n[\mathcal{M}]$ to $\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}$, where $\overline{\otimes}$ is the weak-$*$ closed spatial tensor or equivalently, since $\mathcal{B}(\ell_2)$ is a von Neumann algebra, the Fubini tensor product. The second is a stability result for weak-$*$ closed left ideals in $\mathcal{M}\otimes_{eh}\mathcal{M}$. In the forthcoming text we are going to denote by $\mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M}\otimes_{eh}\mathcal{M})$ the weak-$*$ closed tensor product, with respect to the $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(H))$ topology. Recall that, using the following identifications

$$\begin{aligned}\mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M}\otimes_{eh}\mathcal{M}) &\cong \mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{CB}^{\sigma}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H))\\ &\cong \mathcal{CB}^{\sigma}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H),\mathcal{B}(\ell_2\otimes_2 H))\end{aligned}$$

and reasoning like in (5.1.7), we have that the predual of $\mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M}\otimes_{eh}\mathcal{M})$ can be expressed as a quotient of $\mathcal{B}(H)\,\widehat{\otimes}\,S_1(\ell_2\otimes_2 H)$.

Lemma 5.2.1. *The bilinear map $\odot:\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}\times\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}\to\mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M}\otimes_{eh}\mathcal{M})$ is bounded and continuous over bounded sets if $\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}\times\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}$ is given the product strong operator topology* (SOT) *and $\mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M}\otimes_{eh}\mathcal{M})$ the $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(H))$ topology.*

Proof. Let $(y_\alpha)_\alpha,(x_\alpha)_\alpha\subset\mathrm{Ball}(\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M})$ be nets in the unit ball satisfying that $x_\alpha\to x$ and $y_\alpha\to y$ in the SOT. Since the SOT and σ-SOT topologies agree on bounded set we can assume that we have SOT convergence for any given representation of $\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}$ and in particular for its representation on the Hilbert-Schmidt operators $S_2(\ell_2\otimes_2 H)$. Again since the weak-$*$ topology and the pointwise weak-$*$ topology of $\mathcal{CB}^{\sigma}_{\mathcal{M}'\,\mathcal{M}'}(\mathcal{B}(H),\mathcal{B}(\ell_2\otimes_2 H))$ agree on bounded sets it is enough to see that for any $S\in\mathcal{B}(H)$ and $\xi\in S_1(\ell_2\otimes_2 H)$, $\langle(S\otimes\xi),x_\alpha\odot y_\alpha\rangle\to\langle(S\otimes\xi),x\odot y\rangle$. But using that $\langle(S\otimes\xi),x\odot y\rangle=\langle\xi,x\,(\mathbf{1}\otimes S)\,y\rangle$ and expressing $\xi=\eta\,\zeta^*$, where η,ζ are Hilbert-Schmidt operators, gives $\langle(S\otimes\xi),x\odot y\rangle=\langle\eta,x\,(\mathbf{1}\otimes S)\,y\,\zeta\rangle$, where the last paring is just the inner product of $S_2(\ell_2\otimes_2 H)$. Using the SOT-convergence of x_α and y_α gives

$$\begin{aligned}&|\langle\eta,x_\alpha\,(\mathbf{1}\otimes S)\,y_\alpha-x\,(\mathbf{1}\otimes S)\,y\,\zeta\rangle|\\ \leq\;&|\langle\eta,x_\alpha\,(\mathbf{1}\otimes S)\,(y_\alpha-y)\,\zeta\rangle|+|\langle\eta,(x_\alpha-x)\,(\mathbf{1}\otimes S)\,y\,\zeta\rangle|\\ \leq\;&\Big(\sup_\alpha\|(\mathbf{1}\otimes S^*)\,x_\alpha^*\,\eta\|\Big)\,\|(y_\alpha-y)\,\zeta\|+\|\eta\|\,\|(x_\alpha-x)\,(\mathbf{1}\otimes S)\,y\,\zeta\|\\ \to\;&0,\end{aligned}$$

and that concludes the proof. $\square$

Lemma 5.2.2. *Let $J\subset\mathcal{M}\otimes_{eh}\mathcal{M}$ be a $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(H))$-closed left ideal, the following holds*

(i) *If $X,Y\in\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}$ satisfy that $X\odot Y\in\mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ then $ZX\odot YT\in\mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ for every $Z,T\in\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}$.*

(ii) *$X\odot Y\in\mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ if and only if $[X^*]\odot[Y]\in\mathcal{B}(\ell_2)\,\overline{\otimes}\,J$.*

Proof. Since $J\subset\mathcal{M}\otimes_{eh}\mathcal{M}$ is a $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(H))$-closed left ideal, $J'=\mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ is also a $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(\ell_2\otimes_2 H))$-closed left ideal. Furthermore, it satisfies that $J'(\mathcal{B}(\ell_2)\otimes\mathbf{1})=J'$. For (i) just notice that if $Z=A\otimes x$, $T=B\otimes y$ are simple tensors, then

$$ZX\odot YT=(A\otimes x\otimes y)\,(X\odot Y)\,(B\otimes\mathbf{1})\in J'$$

Now, approximating T and Z by bounded, SOT-convergent nets of sums of simple tensor and applying 5.2.1 we obtain (i).

For (ii) notice that if $[X^*]\odot[Y]\in J'$ then, by (i), $X\,[X^*]\odot[Y]\,Y=X\odot Y\in J'$. For the other implication we just use functional calculus. Indeed, if $X\odot Y\in J'$ then $X^*X\odot YY^*\in J'$.

Let us denote by $P = X^*X$ and $Q = YY^*$ and let $p_n(r)$ be a family of polynomials converging pointwise and boundedly to $\chi_{[0,\infty)}(r)$. Then, since all of the powers $P^n \odot Q^n$ lie in J' we have that $p_n(P) \odot p_n(Q) \in J'$. Since $p_n(P) \to \chi_{[0,\infty)}(P) = [X^*]$ and $p_n(Q) \to \chi_{[0,\infty)}(Q) = [Y]$ in the SOT, we obtain the claim. $\square$

We can now prove the main theorem of the section.

Theorem 5.2.3. *Let $\mathcal{M} \subset \mathcal{B}(H)$ be a von Neumann algebra. The maps*

$$\begin{tikzcd}
\left\{\begin{array}{c}\mathcal{R}\subset\mathcal{P}(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{M})^2: \\ \mathcal{R} \textit{ is an i.q.r.}\end{array}\right\} \arrow[d, bend right, "J_{\mathcal{R}}"'] \arrow[dd, bend right=70, "\mathcal{V}_{\mathcal{R}}"'] \\
\left\{\begin{array}{c}J\subset\mathcal{M}\otimes_{eh}\mathcal{M}: \\ (\mathcal{M}\otimes_{eh}\mathcal{M})\, J\subset J, \\ \sigma(\mathcal{B}\widehat{\otimes} S_1)\textit{-closed}\end{array}\right\} \arrow[u, bend right, "\mathcal{R}_J"'] \arrow[d, bend right, "\mathcal{V}_J"'] \\
\left\{\begin{array}{c}\mathcal{V}\subset\mathcal{B}(H): \\ \mathcal{M}'\,\mathcal{V}\,\mathcal{M}'\subset\mathcal{V},\ \textit{weak-}{*}\textit{closed}\end{array}\right\} \arrow[u, bend right, "J_{\mathcal{V}}"'] \arrow[uu, bend right=70, "\mathcal{R}_{\mathcal{V}}"']
\end{tikzcd}$$

given by

$$\begin{array}{rcl}
J_{\mathcal{V}} &=& \{s \in \mathcal{M} \otimes_{eh} \mathcal{M} : \Phi_s(T) = 0, \forall T \in \mathcal{V}\} \\
\mathcal{R}_{\mathcal{V}} &=& \{(P,Q) \in \mathcal{P}(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{M})^2 : \exists T \in \mathcal{V}, P(T \otimes \mathbf{1})Q \neq 0\} \\
\mathcal{V}_J &=& \{T \in \mathcal{B}(H) : \Phi_s(T) = 0, \forall s \in J\} \\
\mathcal{R}_J &=& \{(P,Q) \in \mathcal{P}(\mathcal{B}(\ell_2)\overline{\otimes}\mathcal{M})^2 : P \odot Q \notin \mathcal{B}(\ell_2)\overline{\otimes} J\} \\
J_{\mathcal{R}} &=& \overline{\{(\phi \otimes \mathrm{Id})(X \odot Y) : ([X^*],[Y]) \in \mathcal{R}, \phi \in \mathcal{B}(\ell_2)_*\}}^{\mathrm{w}*} \\
\mathcal{V}_{\mathcal{R}} &=& \{T \in \mathcal{B}(\ell_2) : P(T \otimes \mathbf{1})Q = 0, \forall (P,Q) \notin \mathcal{R}\},
\end{array}$$

are well defined, bijective and inverse of each other, i.e.

(i) $\mathcal{V}_{J_{\mathcal{V}}} = \mathcal{V}$ (iii) $\mathcal{R}_{J_{\mathcal{R}}} = \mathcal{R}$ (v) $\mathcal{R}_{\mathcal{V}_{\mathcal{R}}} = \mathcal{R}$

(ii) $J_{\mathcal{V}_J} = J$ (iv) $J_{\mathcal{R}_J} = J$ (vi) $\mathcal{V}_{\mathcal{R}_{\mathcal{V}}} = \mathcal{V}$

Furthermore, the rest of the maps commute, giving

(1) $\mathcal{R}_{J_{\mathcal{V}}} = \mathcal{R}_{\mathcal{V}}$ (3) $\mathcal{V}_{\mathcal{R}_J} = \mathcal{V}_J$ (5) $\mathcal{R}_{\mathcal{V}_J} = \mathcal{R}_J$

(2) $J_{\mathcal{V}_{\mathcal{R}}} = J_{\mathcal{R}}$ (4) $J_{\mathcal{R}_{\mathcal{V}}} = J_{\mathcal{V}}$ (6) $\mathcal{V}_{J_{\mathcal{R}}} = \mathcal{V}_{\mathcal{R}}$.

Proof. The fact that $\mathcal{R}_{\mathcal{V}}$ and $\mathcal{V}_{\mathcal{R}}$ are intrinsic quantum relations and weak-$*$ closed $\mathcal{M}'$-bimodules is trivial. Points (v) and (vi) are the content of [Wea12, Theorem 2.32]. We shall prove only the rest of the points.

Proof of (i). $\mathcal{V}_J$ is a weak-$*$ closed $\mathcal{M}'$-bimodule since it is the intersection of $\{T \in \mathcal{B}(H) : \Phi_s(T) = 0\}$ for every $s \in J$ and each of these subspaces is weak-$*$ closed and $\mathcal{M}'$-bimodular. It is also clear that $\mathcal{V} \subset \mathcal{V}_{J_{\mathcal{V}}}$, we only need to prove the converse. Let $T \notin \mathcal{V}$. Since $\mathcal{V} \subset \mathcal{B}(H)$ is weak-$*$ closed there is, by the Hahn-Banach Theorem, a weak-$*$ continuous functional $\phi : \mathcal{B}(H) \to \mathbb{C}$ such that $\phi(S) = 0, \forall S \in \mathcal{V}$ but $\phi(T) \neq 0$. Any such functional is of the form $\phi(A) = \langle \eta, (\mathbf{1} \otimes A)\xi \rangle$, where $\eta, \xi \in \ell_2 \otimes_2 H$. Since $\mathcal{V}$ is an $\mathcal{M}'$-bimodule we have that

$$\langle (\mathbf{1} \otimes x)\, \eta, (\mathbf{1} \otimes S)\, (\mathbf{1} \otimes y)\, \xi \rangle = 0,$$

where $S \in \mathcal{V}$ and $x, y \in \mathcal{M}'$. Let P and Q be the orthogonal projections onto the subspaces of $\ell_2 \otimes_2 H$ given by

$$H_1 = \overline{(\mathbf{1} \otimes \mathcal{M}')\, \eta}, \qquad H_2 = \overline{(\mathbf{1} \otimes \mathcal{M}')\, \xi}.$$

These subspaces are $(\mathbf{1} \otimes \mathcal{M}')$-invariant, therefore $P, Q \in (\mathbf{1} \otimes \mathcal{M})' = \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}$. Clearly we have $P(\mathbf{1} \otimes \mathcal{V})Q = \{0\}$ but $P(\mathbf{1} \otimes T)Q \neq 0$. Let us write $P = [p_{ij}]_{i,j}$ and $Q = [q_{ij}]_{ij}$, where $p_{ij}, q_{ij} \in \mathcal{M}$. Notice that:

$$P(\mathbf{1} \otimes T)Q = \left[\Phi_{r_{ij}}(T)\right]_{ij},$$

where

$$r_{ij} = \sum_{k=1}^{\infty} p_{ik} \otimes q_{kj} \in \mathcal{M} \otimes_{eh} \mathcal{M}.$$

Since $P(\mathbf{1} \otimes T)Q \neq 0$ there are i, j such that $\Phi_{r_{ij}}(T) \neq 0$ but $r_{ij} \in J_{\mathcal{V}}$ which implies that $T \notin \mathcal{V}_{J_{\mathcal{V}}}$ and so $\mathcal{V}_{J_{\mathcal{V}}} \subset \mathcal{V}$, which concludes (i).

Proof of (ii). First, let us see that $J_{\mathcal{V}}$ is $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$-closed. Observe that $\Phi_s(T) = 0$ iff $\langle \xi, \Phi_s(T) \rangle = 0$ for every $\xi \in \mathcal{B}(H)_*$. Therefore

$$\{s \in \mathcal{M} \otimes_{eh} \mathcal{M} : \Phi_s(T) = 0\} = \bigcap_{\xi \in S_1(H)} \{s \in \mathcal{M} \otimes_{eh} \mathcal{M} : \langle \xi, \Phi_s(T) \rangle = 0\},$$

and so, the left hand side is pointwise weak-$*$ closed. Since the $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$ topology is finer than the pointwise weak-$*$ topology of $\mathcal{CB}^{\sigma}_{\mathcal{M}', \mathcal{M}'}(\mathcal{B}(H))$ we have that $\{s \in J : \Phi_s|_{\mathcal{V}} = 0\}$ is a weak-$*$ closed subspace. The fact that it is a left ideal follows trivially from the definition.

Let J be a $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$-closed left ideal. Again, it is clear that $J \subset J_{\mathcal{V}_J}$ we only have to prove the other containment. That is equivalent to prove that for every $s_0 \notin J$ there is $T \in \mathcal{B}(H)$ such that $\Phi_s(T) = 0$ for every $s \in J$ and $\Phi_{s_0} \neq 0$. By weak-$*$ closeness of J and the Hahn-Banach theorem there is a weak-$*$ continuous functional $\phi \in (\mathcal{M} \otimes_{eh} \mathcal{M})_*$ such that $\langle \phi, s \rangle = 0$ for every $s \in J$ but $\langle \phi, s_0 \rangle \neq 0$. By Lemma 5.1.5 we have that

$$\langle \phi, s \rangle = \big\langle C, (\mathrm{Id} \otimes \Phi_s)(B) \big\rangle,$$

where $C \in S_1(\ell_2 \otimes_2 H)$ and $B \in \mathcal{B}(\ell_2 \otimes_2 H)$. We can decompose $C = C_1\, C_2^*$ where $C_1, C_2 \in S_2(\ell_2 \otimes_2 H)$ and so

$$\langle \phi, s \rangle = \big\langle C_1, (\mathrm{Id} \otimes \Phi_s)(B)\, C_2 \big\rangle$$

where $(\mathrm{Id} \otimes \Phi_s)(B) \in \mathcal{B}(\ell_2 \otimes_2 H)$ is acting on $S_2(\ell_2 \otimes_2 H)$ by left multiplication and the duality pairing is that of S_2 with itself. We have that, for every $s \in J$, $\langle \phi, s \rangle = 0$, and so, since J is an ideal, $\langle \phi, (x \otimes y)\, s \rangle = 0$. Therefore

$$0 = \big\langle (\mathbf{1} \otimes x^*)\, C_1, (\mathrm{Id} \otimes \Phi_s)(B)\, (\mathbf{1} \otimes y)\, C_2 \big\rangle. \tag{5.2.1}$$

Let us define the closed subspaces $H_1, H_2 \subset S_2(\ell_2 \otimes_2 H)$ given by

$$H_1 = \overline{(1 \otimes \mathcal{M})\, C_1}, \qquad H_2 = \overline{(1 \otimes \mathcal{M})\, C_2}$$

and let $P_i : S_2(\ell_2 \otimes_2 H) \to H_i$, for $i \in \{1, 2\}$, be their orthogonal projections. We can identify isometrically $S_2(\ell_2 \otimes_2 H) \cong \ell_2 \otimes_2 H \otimes_2 \ell_2 \otimes_2 H$, such identification gives that $\mathcal{B}(S_2(\ell_2 \otimes_2 H)) \cong$

$\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H) \overline{\otimes} \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H)$, where the first two components correspond to right multiplication and the other two correspond to left multiplication. Since H_1 and H_1 are $\mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathcal{M}$-invariant the projections P_1, P_2 belong to to $(\mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1} \otimes \mathcal{M})' = \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H) \overline{\otimes} \mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}'$. Now, the identity 5.2.1 implies that

$$0 = P_1 \, (\mathrm{Id} \otimes \Phi_s)(B) \, P_2,$$

where $(\mathrm{Id} \otimes \Phi_s)(B)$ is seen as an operator in $\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{B}(H) \otimes \mathbb{C}\mathbf{1} \otimes \mathbb{C}\mathbf{1}$. If $s = \sum_k x_k \otimes y_k$ we have that

$$\begin{aligned} P_1 \, (\mathrm{Id} \otimes \Phi_s)(B) \, P_2 &= \sum_k P_1 \, (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes x_k) \, B \, (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes y_k) \, P_2 \\ &= \sum_k (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes x_k) \, (P_1 \, B \, P_2) \, (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes y_k) \\ &= (\mathrm{Id}_{\mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)} \otimes \Phi_s)(P_1 \, B \, P_2). \end{aligned}$$

Let $T_\xi \in \mathcal{B}(H)$ be the operator given by $(\xi \otimes \mathrm{Id}_{\mathcal{B}(H)})(P_1 \, B \, P_1) \in \mathcal{B}(H)$, where $\xi \in \mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)_*$. We have that $\Phi_s(T_\xi) = 0$ for every $s \in J$ since

$$\begin{aligned} \Phi_s(T_\xi) &= (\xi \otimes \Phi_s)(P_1 \, B \, P_1) \\ &= \langle \xi, (\mathrm{Id}_{\mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)} \otimes \Phi_s)(P_1 \, B \, P_2) \rangle \\ &= 0. \end{aligned}$$

But there has to be a $\xi_0 \in \mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)_*$ such that $\Phi_{s_0}(T_{\xi_0}) \neq 0$, otherwise

$$\langle \xi, P_1 (\mathrm{Id}_{\mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)} \otimes \Phi_{s_0})(B) P_2 \rangle = 0,$$

for every $\xi \in \mathcal{B}(\ell_2 \otimes_2 H \otimes_2 \ell_2)_*$ which implies that $P_1 (\mathrm{Id} \otimes \Phi_{s_0})(B) P_2 = 0$ but that is impossible since C_1 and C_2 are in the ranges of P_1 and P_2 respectively. The existence of such T_{ξ_0} finishes the proof.

Proof of (iii). We will start proving that $\mathcal{R}_J$ is an intrinsic quantum relation. First, we have to see that $\mathcal{R}_J$ is weak-$*$ open. Since J is $\sigma(\mathcal{B}(H) \widehat{\otimes} S_1(H))$-closed, so is $\mathcal{B}(\ell_2) \overline{\otimes} J$. The complementary $(\mathcal{B}(\ell_2) \overline{\otimes} J)^{\mathrm{c}}$ is weak-$*$ open and so is $\mathcal{R}_J$, since it is the reverse image of $(\mathcal{B}(\ell_2) \overline{\otimes} J)^{\mathrm{c}}$ under the function $\odot : \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}) \times \mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M}) \to \mathcal{B}(\ell_2) \overline{\otimes} (\mathcal{M} \otimes_{eh} \mathcal{M})$, which is weak-$*$ continuous by Lemma 5.2.2 (recall that over $\mathcal{P}(\mathcal{B}(\ell_2) \overline{\otimes} \mathcal{M})$ the SOT, WOT, σ-SOT and σ-WOT coincide). Second, we are going to prove the properties (i), (ii), (ii) in Definition 5.1.1. It is trivial that $(0, 0) \notin \mathcal{R}_J$. For (ii) we have to prove that

$$\forall \alpha, \beta \, , (P_\alpha, Q_\beta) \in \mathcal{B}(\ell_2) \overline{\otimes} J \iff \big(\bigvee_\alpha P_\alpha, \bigvee_\beta Q_\beta \big) \in \mathcal{B}(\ell_2) \overline{\otimes} J.$$

For the implication $(\Longrightarrow)$ we use that if $(P_\alpha, Q_\beta) \in \mathcal{B}(\ell_2) \overline{\otimes} J$ then

$$\big(\sum_\alpha P_\alpha, \sum_\beta Q_\beta \big) \in \mathcal{B}(\ell_2) \overline{\otimes} J,$$

but using that, for any family of projections $(R_\gamma)_\gamma$

$$[\sum_\gamma R_\gamma] = \bigvee_\gamma R_\gamma$$

and Lemma 5.2.2 (ii) we obtain that

$$\big(\bigvee_\alpha P_\alpha, \bigvee_\beta Q_\beta\big) = \big(\big[\sum_\alpha P_\alpha\big], \big[\sum_\beta Q_\beta\big]\big) \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J.$$

Proving ($\Longleftarrow$) is clearly equivalent to proving that $P\odot Q \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ implies that $R\odot S \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ for any projections $R \leq P$ and $S \leq Q$, but that follows trivially from Lemma 5.2.2 (i). For point (iv) we have that if $P \odot [BQ] \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ then $P \odot BQ \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J$ by Lemma 5.2.2. Since $B \in \mathcal{B}(\ell_2) \otimes \mathbb{C}\mathbf{1}$ we have that $P \odot BQ = PB \odot Q \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J$, again by Lemma 5.2.2, that implies that $[B^*P] \odot Q \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,J$. The other implication is proved similarly.

In order to prove the inclusion $\mathcal{R}_{J_\mathcal{R}} \subset \mathcal{R}$ start by noticing that:

$$\begin{aligned}
\mathcal{B}(\ell_2)\,\overline{\otimes}\,J_\mathcal{R} &= \mathcal{B}(\ell_2)\,\overline{\otimes}\,\overline{\{(\phi\otimes \mathrm{Id})(X\odot Y) : \phi \in \mathcal{B}(\ell_2)_*, ([X^*],[Y]) \notin \mathcal{R}\}}^{\mathrm{w}^*} \\
&= \overline{\mathrm{span}}^{\mathrm{w}^*}\{X \odot Y : ([X^*],[Y]) \notin \mathcal{R}\}.
\end{aligned}$$

If we assume that $P\odot Q \notin \overline{\mathrm{span}}^{\mathrm{w}^*}\{X\odot Y : ([X^*],[Y]) \notin \mathcal{R}\}$ then trivially we have that $(P,Q) \in \mathcal{R}$. For the other inclusion we shall use that, by (v), $(P,Q) \in \mathcal{R}$ iff $(P,Q) \in \mathcal{R}_{\mathcal{V}_\mathcal{R}}$ which happens only when $P\,(\mathbf{1}\otimes A)\,Q \neq 0$ for some $A \in \mathcal{V}_\mathcal{R}$. Since the complete isometry $\mathrm{Id}\otimes\Phi : \mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M}\otimes_{eh}\mathcal{M}) \to \mathcal{CB}^\sigma_{\mathcal{M}',\mathcal{M}'}(\mathcal{B}(H), \mathcal{B}(\ell_2 \otimes_2 H))$ satisfies that $(\mathrm{Id}\otimes\Phi)_{X\odot Y}(A) = X\,(\mathbf{1}\otimes A)\,Y$ we have that

$$\begin{aligned}
\overline{\mathrm{span}}^{\mathrm{w}^*}\{X\odot Y : ([X^*],[Y]) \notin \mathcal{R}\} &= \overline{\mathrm{span}}^{\mathrm{w}^*}\{X\odot Y : [X^*]\,(\mathbf{1}\otimes\mathcal{V}_\mathcal{R})\,[Y] = \{0\}\} \\
&= \overline{\mathrm{span}}^{\mathrm{w}^*}\{X\odot Y : (\mathrm{Id}\otimes\Phi)_{[X^*]\odot[Y]}|_{\mathcal{V}_R} = 0\} \\
&\subset \{s : (\mathrm{Id}\otimes\Phi)_s|_{\mathcal{V}_\mathcal{R}} = 0\}.
\end{aligned}$$

But no pair $(P,Q) \in \mathcal{R}$ satisfies that $P\odot Q \in \{X\odot Y : ([X^*],[Y]) \notin \mathcal{R}\}$ since that will imply that $(\mathrm{Id}\otimes\Phi)_{P\odot Q}|_{\mathcal{V}_\mathcal{R}} = 0$ and that is a contradiction.

Proof of (iv). Let us see that $J_\mathcal{R}$ is an ideal for every intrinsic quantum relation $\mathcal{R}$. To see that it is a linear subspace fix $\phi_1, \phi_2 \in \mathcal{B}(\ell_2)_*$ and $([X_1^*],[Y_1]) \notin \mathcal{R}$, $([X_2^*],[Y_2]) \notin \mathcal{R}$. If $B_1 : \ell_2 \to \ell_2$ and $B_2 : \ell_2 \to \ell_2$ are isometries whose ranges are orthogonal and complementary. We have that the operators

$$\begin{aligned}
X &= (B_1\otimes\mathbf{1})\,X_1\,(B_1^*\otimes\mathbf{1}) + (B_2\otimes\mathbf{1})\,X_2\,(B_1^*\otimes\mathbf{1}) \\
Y &= (B_1\otimes\mathbf{1})\,Y_1\,(B_1^*\otimes\mathbf{1}) + (B_2\otimes\mathbf{1})\,Y_2\,(B_1^*\otimes\mathbf{1})
\end{aligned}$$

satisfy $[X^*] = (B_1\otimes\mathbf{1})\,[X_1^*]\,(B_1^*\otimes\mathbf{1}) + (B_2^*\otimes\mathbf{1})\,[X_2^*]\,(B_1^*\otimes\mathbf{1})$, $[Y] = (B_1\otimes\mathbf{1})\,[Y_1]\,(B_1^*\otimes\mathbf{1}) + (B_2^*\otimes\mathbf{1})\,[Y_2]\,(B_1^*\otimes\mathbf{1})$ and therefore, by [KW12, Lemma 2.29], $([X^*],[Y]) \in \mathcal{R}$. Now, a trivial calculation gives that

$$\begin{aligned}
& (\phi_1\otimes\mathrm{Id})(X_1\odot Y_1) + (\phi_2\otimes\mathrm{Id})(X_2\odot Y_2) \\
= \; & \big((B_1\,\phi_1\,B_1^* + B_2\,\phi_2\,B_2^*)\otimes\mathrm{Id}\big)(X\odot Y),
\end{aligned}$$

where $B_i\,\phi_j\,B_i^*(x) = \phi_j(B_i^*\,x\,B_i)$. The fact that $J_\mathcal{R}$ is closed by scalar multiplication is trivial. It is also $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(H))$-closed by construction. It only rest to see that it is absorbent for the multiplication. It is enough to prove that $(z\otimes t)\,J_\mathcal{R} \subset J_\mathcal{R}$ for every $z, t \in \mathcal{M}$. We have that

$$(z\otimes t)\,(\phi\otimes\mathrm{Id})(X\odot Y) = (\phi\otimes\mathrm{Id})((\mathbf{1}\otimes z)\,X\odot Y\,(\mathbf{1}\otimes t)).$$

Now, using that $[Y\,(\mathbf{1}\otimes t)] \leq [Y]$ and $[X^*\,(\mathbf{1}\otimes z^*)] \leq [X^*]$ and applying point (ii) in Definition 5.1.1 gives the desired result.

The inclusion $J_{\mathcal{R}_J} \subset J$ is easy to prove. Recall that if $s \in \mathcal{B}(\ell_2)\,\overline{\otimes}\, J$ then $(\phi \otimes \mathrm{Id})(s) \in J$. Using that together with Lemma 5.2.2(ii) gives

$$\begin{array}{rl} & \overline{\{(\phi \otimes \mathrm{Id})(X \odot Y) : ([X^*],[Y]) \notin \mathcal{R}_J, \phi \in \mathcal{B}(\ell_2)_*\}}^{\mathrm{w}^*} \\ = & \overline{\{(\phi \otimes \mathrm{Id})(X \odot Y) : [X^*] \odot [Y] \in \mathcal{B}(\ell_2)\,\overline{\otimes}\, J, \phi \in \mathcal{B}(\ell_2)_*\}}^{\mathrm{w}^*} \\ \subset & J. \end{array}$$

For the reciprocal inclusion $J \subset J_{\mathcal{R}_J}$ we need to see that if $s \in J$ then there are $\phi \in \mathcal{B}(\ell_2)_*$, $X, Y \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}$ with $([X^*],[Y]) \notin \mathcal{R}_J$ such that $s = (\phi \otimes \mathrm{Id})(X \odot Y)$. Note that we can express

$$s = \sum_{k=0}^{\infty} x_k \otimes y_k = (\omega_{e_1,e_1} \otimes \mathrm{Id})(X \odot Y),$$

as

$$X = \sum_{k=0}^{\infty} x_k \otimes e_{1\,k} \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M} \quad \text{and} \quad Y = \sum_{k=0}^{\infty} y_k \otimes e_{k\,1} \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M}.$$

We only have to prove that $([X^*],[Y]) \notin \mathcal{R}_J$, i.e. that $[X^*] \odot [Y] \in \mathcal{B}(\ell_2)\,\overline{\otimes}\, J$. Again, by Lemma 5.2.2(ii), we only have to see that $X \odot Y \in \mathcal{B}(\ell_2)\,\overline{\otimes}\, J$, which is equivalent to see that for every $\phi \in \mathcal{B}(\ell_2)_*$, $(\phi \otimes \mathrm{Id})(X \odot Y) \in J$. Notice that if P is the projection on the 1-dimensional subspace spanned by e_1, then $X \odot Y = (P \otimes \mathbf{1})\,(X \odot Y)\,(P \otimes \mathbf{1})$. Therefore $(\phi \otimes \mathrm{Id})(X \odot Y) = (P\,\phi\,P \otimes \mathrm{Id})(X \odot Y) = (\lambda\,\omega_{e_1,e_1} \otimes \mathrm{Id})(X \odot Y) = \lambda\, s \in J$, for some $\lambda \in \mathbb{C}$. That finishes the proof of (iv).

Since we have already proved (i)-(vi) we have that (4)-(6) can be deduced from (1)-(3). We will prove only those first three cases, which are easy after the previous results.

Proof of (1). We have that

$$\mathcal{R}_{J_\mathcal{V}} = \{(P,Q) \in \mathcal{P}(\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M})^2 : P \odot Q \notin \mathcal{B}(\ell_2)\,\overline{\otimes}\, J_\mathcal{V}\}$$

and that

$$\mathcal{B}(\ell_2)\,\overline{\otimes}\, J_\mathcal{V} = \{s \in \mathcal{B}(\ell_2)\,\overline{\otimes}\,(\mathcal{M} \otimes_{eh} \mathcal{M}) : (\mathrm{Id} \otimes \Phi)_s|_\mathcal{V} = 0\},$$

therefore

$$\begin{array}{rcl} \mathcal{R}_{J_\mathcal{V}} & = & \{(P,Q) \in \mathcal{P}(\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M})^2 : (\mathrm{Id} \otimes \Phi)_{P \odot Q}|_\mathcal{V} \neq 0\} \\ & = & \{(P,Q) \in \mathcal{P}(\mathcal{B}(\ell_2)\,\overline{\otimes}\,\mathcal{M})^2 : P(\mathbf{1} \otimes \mathcal{V})Q \neq \{0\}\} \quad = \quad \mathcal{R}_\mathcal{V}. \end{array}$$

Proof of (2). Let us start by seeing that $J_\mathcal{R} \subset J_{\mathcal{V}_\mathcal{R}}$. If $z = (\phi \otimes \mathrm{Id})(X \odot Y)$, with $([X^*],[Y]) \notin \mathcal{R}$, then $(\mathrm{Id} \otimes \Phi)_z(T) = (\phi \otimes \mathrm{Id})(X\,(\mathbf{1} \otimes T)\,Y) = (\phi \otimes \mathrm{Id})(X\,[X^*]\,(\mathbf{1} \otimes T)\,[Y^*]\,Y)$. So if $T \in \mathcal{V}_\mathcal{R}$ then $(\mathrm{Id} \otimes \Phi)_z(T) = 0$. For the converse inclusion let $s \in J_{\mathcal{V}_\mathcal{R}}$ and express s as $s = (\omega_{e_1,e_1} \otimes \mathrm{Id})(X \odot Y)$ like in the proof of (iv). We have that $X\,(\mathbf{1} \otimes \mathcal{V}_\mathcal{R})\,Y = 0$ and so $[X^*]\,(\mathbf{1} \otimes \mathcal{V}_\mathcal{R})\,[Y] = 0$ which implies that $([X^*],[Y]) \notin \mathcal{R}_{\mathcal{V}_\mathcal{R}} = \mathcal{R}$ and so $s \in J_\mathcal{R}$.

Proof of (3). The inclusion $\mathcal{V}_J \subset \mathcal{V}_{\mathcal{R}_J}$ is trivial. In order to prove the converse, $\mathcal{V}_{\mathcal{R}_J} \subset \mathcal{V}_J$, fix $S \in \mathcal{V}_{\mathcal{R}_J}$. We have that $X\,(\mathbf{1} \otimes S)\,Y = 0$, $\forall\,([X^*],[Y])$ such that $[X^*] \odot [Y] \in \mathcal{B}(\ell_2)\,\overline{\otimes}\, J$. Then, for any $\phi \in \mathcal{B}(\ell_2)_*$ we have that

$$0 = (\phi \otimes \mathrm{Id})(X\,(\mathbf{1} \otimes S)\,Y) = (\phi \otimes \mathrm{Id})((\mathrm{Id} \otimes \Phi)_{X \odot Y}(S)) = \Phi_{(\phi \otimes \mathrm{Id})(X \odot Y)}(S).$$

Therefore $\Phi_z(S) = 0$ for every $z \in J_{\mathcal{R}_J} = J$. $\square$

Recall that the technique of the proof of point (i) follows exactly the same lines of [Wea12, Lemma 2.8].

Remark 5.2.4. Observe that, a priori, it is not clear why all $\sigma(\mathcal{B}(H)\,\widehat{\otimes}\,S_1(H))$-closed ideals are closed in the coarser pointwise weak-$*$ topology. Such result is obtained as a consequence from Theorem 5.2.3.(ii).

5.3 Invariant Quantum Relations

Let $\mathcal{A}$ be a von Neumann algebraic *quantum group* with comultiplication Δ, see [VD14] for a precise definition, we will say that $\mathcal{M}$ is a *quantum homogeneous space* if there is a normal, $*$-homomorphism $\sigma : \mathcal{M} \to \mathcal{A}\,\overline{\otimes}\,\mathcal{M}$, called the *coaction*, satisfying the natural coassociativity identity

$$(\mathrm{Id} \otimes \sigma)\,\sigma = (\Delta \otimes \mathrm{Id})\,\sigma$$

If $\mathcal{M} \subset \mathcal{B}(H)$ is an *standard form* for the von Neumann algebra $\mathcal{M}$, we have that, after endowing H with its row (resp. column) operator space structure, the coaction σ extends to a complete isometry $\sigma_2 : H^{\mathrm{r}} \to \mathcal{A}\,\overline{\otimes}\,H^{\mathrm{r}}$. We will say that an operator $T \in \mathcal{B}(H)$ is *σ-equivariant* iff

$$\sigma_2\,T = (\mathrm{Id} \otimes T)\,\sigma_2$$

and we will denote by $\mathcal{B}(H)^{\sigma}$ the space of σ-equivariat operators. Similarly, we say that a quantum relation $\mathcal{V}$ over $\mathcal{M}$ is *σ-invariant*, or simply *invariant* if the coaction is understood from the context, iff it is generated (as an operator $\mathcal{M}'$-bimodule) by σ-equivariant operators. If $\mathcal{V}$ is generated by equivariant operators, then it is generated by the equivariant operators inside $\mathcal{V}$, therefore $\mathcal{V}$ is invariant iff

$$\mathcal{V} = {}_{\mathcal{M}'}\,\langle \mathcal{V} \cap \mathcal{B}(H)^{\sigma} \rangle_{\mathcal{M}'} = \overline{\mathrm{span}}^{\mathrm{w}^*}\{x\,T\,y : x, y \in \mathcal{M}', \quad T \in \mathcal{V} \cap \mathcal{B}(H)^{\sigma}\}.$$

From now on we will denote $\mathcal{V} \cap \mathcal{B}(H)^{\sigma}$ by $\mathcal{V}^{\sigma}$. Our purpose in this section is to study invariant quantum relations. Interesting examples of quantum homogeneous spaces include, among others, the ones listed bellow.

Classical homogeneous spaces Let G is a locally compact Hausdorff group and X be a measurable G-space. $L_\infty(G)$ is clearly a quantum group with the comultiplication given by $\Delta(f)(g,h) = f(g\,h)$. Similarly, we can define the coaction $\sigma : L_\infty(X) \to L_\infty(G)\,\overline{\otimes}\,L_\infty(X)$ given by $\sigma(f)(g,x) = f(g\,x)$. To solidify our intuition let us see what happens when X is discrete. In that case quantum relations over $L_\infty(X)$ are just subsets $R \subset X \times X$. Recall that a classical relation $R \subset X \times X$ is G-invariant iff

$$(x,y) \in R \Longleftrightarrow (g\,x, g\,y) \in R,\ \forall g \in G. \tag{5.3.1}$$

We are going to see that such relations correspond with σ-invariant quantum relations. An operator $T = [a_{x\,y}]_{x,y\in X} \in \mathcal{B}(L_2X)$ is σ-equivariant iff it commutes with the action $\sigma_g(f)(x) = f(g^{-1}\,x)$, therefore the set

$$R_T = \{(x,y) \in X \times X : \langle e_x, Te_y\rangle \neq 0\} \subset X \times X$$

satisfies (5.3.1) and the same goes for $R_{\mathcal{V}}$, where $\mathcal{V} =_{\mathcal{R}G} \langle \mathcal{V}^{\sigma} \rangle_{\mathcal{R}G}$, since

$$R_{\mathcal{V}} = \bigcup_{T \in \mathcal{V}^{\sigma}} \{(x,y) \in X \times X : \langle e_x, Te_y\rangle \neq 0\}.$$

This proves that any σ-invariant quantum relation over a discrete space X corresponds to an invariant relation $R \subset X \times X$. The reciprocal is shown similarly.

Group von Neumann algebras Let G be a locally compact Hausdorff group, $L_2(G)$ the L_2-space with respect to the left Haar measure and $\lambda : G \to \mathcal{U}(L_2G)$ be the unitary representation given by $\lambda_g(\xi)(h) = \xi(g^{-1}h)$, where $\xi \in L_2(G)$. The (left) group von Neumann algebra $\mathcal{L}G$ is given by

$$\mathcal{L}G = \{\lambda_g : g \in G\}'' \subset \mathcal{B}(L_2G).$$

The natural comultiplication structure $\Delta : \mathcal{L}G \to \mathcal{L}G \otimes_{eh} \mathcal{L}G$ is given by $\lambda_g \mapsto \lambda_g \otimes \lambda_g$. In this case the commutant $\mathcal{L}G'$ is given by the (right) group von Neumann algebra

$$\mathcal{R}G = \{\rho_g : g \in G\}'' \subset \mathcal{B}(L_2G)$$

where $\rho_g(\xi)(h) = \xi(h\,g)\,\Delta(g)^{\frac{1}{2}}$, where $\xi \in L_2(G)$ is right regular representation. We can considere $\mathcal{L}G$ a quantum homogeneous space over itself with the multiplication as coaction. The representation $\mathcal{L}G \subset \mathcal{B}(L_2G)$ is standard, and the Δ-equivariant operators are given by the subalgebra $L_\infty(G)$ acting by multiplication operation. It is also illustrative to observe that if we take the GNS representation associated with the canonical Plancherel weight φ, see [Ped79], $\mathcal{L}G \subset \mathcal{B}(L_2(\mathcal{L}G, \varphi))$, then an element $T : L_2(\mathcal{L}G, \varphi) \to L_2(\mathcal{L}G, \varphi)$ is Δ-equivariant iff it is a noncommutative Fourier multiplier over $L_2(\mathcal{L}G)$, in the sense of [CdlS15, 3.7]. By the Plancherel theorem, the algebra of such multipliers is equivalent to $L_\infty(G)$.

Quantum Torii One family of von Neumann algebras that has received a considerable amount of attention is that of *quantum torii* $\mathcal{A}_\theta^n \subset L_2(\mathbb{T}^n)$. In such case the coaction is given by $\sigma : \mathcal{A}_\theta^n \to L_\infty(\mathbb{T}^n) \,\overline{\otimes}\, \mathcal{A}_\theta^n$. Quantum relations on quantum torii have been considered before in [Wea12, Section 2.7].

Here, we will mainly focus our attention on the case of $\mathcal{M} = \mathcal{L}G$. Our purpose is to describe the ideals associated with invariant quantum relations over $\mathcal{L}G$. For that, we need to recall some results on the representation of completely bounded $\mathcal{R}G$-bimodular operators preserving the Δ-equivariant operators. Let MG be the Banach algebra of finite measures with the o.s.s. given by $C_0(G)^* = MG$. Apart from the weak-$*$ topology given by $\sigma(C_0G)$ in MG we have the strictly finer topology generated by evaluation against every bounded continuous function $\sigma(C_bG)$. Reasoning like before, since MG is $\sigma(C_bG)$-closed, the $\sigma(C_bG)$ topology induces another predual for MG. The subalgebra of point measures $\ell_1(G) \subset MG$ is of course $\sigma(C_bG)$-dense. We define a multiplicative and injective map $j : \ell_1(G) \to \mathcal{L}G \otimes_{eh} \mathcal{L}G$ by $\delta_g \mapsto \lambda_g \otimes \lambda_{g^{-1}}$. The following theorem assure that there is an injective and weak-$*$ continuous extension to MG and characterizes its range as normal $\mathcal{R}G$-bimodular, c.b. maps preserving $\mathcal{B}(L_2G)^\Delta = L_\infty(G)$. We will denote the algebra of such operators by $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2G)) \subset \mathcal{CB}^{\sigma}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2G))$. Such algebra is closed, with respect to the natural weak-$*$ topologies of $\mathcal{CB}^{\sigma}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2G))$ and so it inherits both the $\sigma(\mathcal{B}(L_2G) \,\widehat{\otimes}\, S_1(L_2G))$ and the $\sigma(\mathcal{K}(L_2G) \,\widehat{\otimes}\, S_1(L_2G))$ topologies.

Theorem 5.3.1 ([NRS08, Theorem 3.2]). *Let G be a locally compact group. There is a $\sigma(C_b)$ to $\sigma(\mathcal{B} \,\widehat{\otimes}\, S_1)$ continuous, multiplicative and injective complete isometry $j : MG \to \mathcal{L}G \otimes_{eh} \mathcal{L}G$ extending the map $\delta_g \mapsto \lambda_g \otimes \lambda_{g^{-1}}$. Furthermore, the following diagram commute*

$$\begin{tikzcd}
MG \arrow[r, hook, "j"] \arrow[d, "\Theta"] & \mathcal{L}G \otimes_{eh} \mathcal{L}G \arrow[d, "\Phi"] \\
\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(H)) \arrow[r, hook, "\subseteq"] & \mathcal{CB}^{\sigma}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(H))
\end{tikzcd}$$

In particular, $\Theta : MG \to \mathcal{CB}^{\sigma}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2G))$ is a complete isometry whose range is $\mathcal{CB}^{\sigma, L_\infty(G)}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2G))$.

The topology induced by $\sigma(\mathcal{K} \widehat{\otimes} S_1)$ in MG is the just $\sigma(C_0)$, while the topology induced by $\sigma(\mathcal{B} \widehat{\otimes} S_1)$ is $\sigma(C_b)$.

We will, perhaps ambiguously, denote by Θ either the map $\Theta : MG \to \mathcal{CB}^{\sigma}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2 G))$ or the restriction to its image.

We will briefly sketch the proof of the theorem above since some of its ideas will be used in the forthcoming results. But, before that, we need to recall a few well known facts on the theory of crossed products. Let $r : G \to \mathrm{Aut}(L_\infty G)$ be the normal right-translation action given by $r_g(f)(x) = f(x\, g)$ noticing that by the Takai-Takesaki duality theorem, see [Tak73], we have

$$L_\infty(G) \rtimes_r G = \mathcal{B}(L_2 G),$$

where $\rtimes$ is notation for the (weak-$*$ closed) spatial crossed product. The action r is spatially implemented on $L_\infty(G) \subset \mathcal{B}(L_2 G)$ by the right regular representation, i.e. $r_g(f) = \rho_g\, f\, \rho_{g^{-1}}$ and so we obtain that

$$L_\infty(G) \rtimes_r G = \{L_\infty(G), \mathcal{R}G\}'' = \mathcal{B}(L_2 G).$$

As a consequence, we can identify $L_\infty(G) \rtimes \mathbf{1} \subset L_\infty(G) \rtimes_r G = \mathcal{B}(L_2 G)$ with the algebra of Δ-equivariant operators.

Proof. First, we are going to see that the map $\Theta : MG \to \mathcal{CB}^{\sigma}_{\mathcal{R}G\text{-}\mathcal{R}G}(\mathcal{B}(L_2 G))$ is surjective. Observe that Θ_μ acts on $L_\infty(G) \subset \mathcal{B}(L_2 G)$ by left convolution, i.e. $\Theta_\mu(f) = \mu * f$. Notice also that, if $\Psi : \mathcal{B}(L_2 G) \to \mathcal{B}(L_2 G)$ is a normal and $\mathcal{R}G$-bimodular map, its restriction $\Psi|_{L_\infty(G)} : L_\infty(G) \to L_\infty(G)$ determines the map Ψ since $L_\infty(G)$ and $\mathcal{R}G$ generate the whole von Neumann algebra $\mathcal{B}(L_2 G)$ by the Takai-Takesaki theorem. Furthermore, since Ψ preserves $L_\infty(G)$, we have that, for every $f \in L_\infty(G)$

$$\rho_g\, \Psi(f) = \Psi(\rho_g\, f) = \Psi(\rho_g\, f \rho_{g^{-1}}\, \rho_g) = \Psi(r_g\, f)\, \rho_g$$

and so $\Psi|_{L_\infty(G)}$ is a right-translation equivariant operator, i.e. $r_g\, \Psi = \Psi\, r_g$. But then, any such operator is actually given by left convolution with a finite measure, see [Wen52]. So $\Psi(f) = \mu * f = \Theta_\mu(f)$ and since Ψ and Θ coincide in $L_\infty(G)$ they are equal.

Reciprocally, if we pick a measure $\mu \in MG$ we have that the map $T_\mu : L_\infty(G) \to L_\infty(G)$ given by $f \mapsto \mu * f$ is a normal bounded operator commuting with r_g for all $g \in G$. Since $L_\infty(G)$ is an abelian operator space we have that T_μ is c.b. and that

$$\|T_\mu\|_{\mathrm{cb}} = \|\mu\|_{MG}.$$

But for any crossed product there is a normal injective $*$-homomorphism $\iota : L_\infty(G) \rtimes G \to L_\infty(G) \overline{\otimes} \mathcal{B}(L_2 G) \subset \mathcal{B}(L_2 G \otimes_2 L_2 G)$. To define such embedding ι fix an element $\xi \in L_2(G \times G)$. The action of $\iota(f\, \rho_{g_0})$ on ξ is given by

$$j(f\, \rho_{g_0})\, \xi(g, h) = f(g\, h^{-1}\, g_0^{-1})\, \xi(g, g_0^{-1}\, h).$$

while for general $x \in L_\infty(G) \rtimes_r G$ we just extend linearly and take weak-$*$ limits. Such embedding appears naturally in the crossed product construction. It satisfies that, if T is equivariant, then

$$\begin{tikzcd} L_\infty(G) \rtimes G \arrow[r, hook, "\iota"] \arrow[d, "T \rtimes \mathrm{Id}"] & L_\infty(G) \overline{\otimes} \mathcal{B}(L_2 G) \arrow[d, "T \otimes \mathrm{Id}"] \\ L_\infty(G) \rtimes G \arrow[r, hook, "\iota"] & L_\infty(G) \overline{\otimes} \mathcal{B}(L_2 G) \end{tikzcd} .$$

As a consequence, if T is completely bounded so is $T \rtimes \mathrm{Id}$ and

$$\|T \rtimes \mathrm{Id}\|_{\mathrm{cb}} = \|T \otimes \mathrm{Id}\|_{\mathrm{cb}}$$

After identifying $L_\infty(G) \rtimes_r G$ with $\mathcal{B}(L_2G)$, we get that $T_\mu \rtimes \mathrm{Id}$ is the only normal and $\mathcal{R}G$-bimodular extension of T_μ. Therefore $\Theta_\mu = T_\mu \rtimes \mathrm{Id}$ and so Θ is well defined and isometric. Since Θ clearly factors through $\mathcal{L}G \otimes_{eh} \mathcal{L}G$ we also obtain that j is a complete isometry. □

The result above goes back to Wendel [Wen52] but the formulation is taken from [NRS08], whose main contribution is to generalize the result from $\mathcal{L}G$ to its quantum group dual $L_\infty(G)$ obtaining a complete isomorphism $\hat{\Theta} : M_{\mathrm{cb}}AG \to \mathcal{CB}^{\sigma,\mathcal{L}G}_{L_\infty(G)\text{-}L_\infty(G)}(\mathcal{B}(L_2G))$, where $M_{\mathrm{cb}}AG$ is the space of completely bounded multipliers of the Fourier algebra AG. It is also worth pointing out that both results can be unified using the language of quantum groups, see [JNR09].

The main result of this section is that invariant quantum relations over $\mathcal{L}G$ are in bijective correspondence with weak-$*$ closed left ideals inside MG.

Theorem 5.3.2. *Let $\mathcal{L}G \subset \mathcal{B}(H)$ be as above. If $\mathcal{V}$ is an invariant quantum relation over $\mathcal{L}G$ and $Q \subset MG$ is a $\sigma(C_bG)$-closed left ideal, then the following maps*

(1) $Q_\mathcal{V} = \{\mu \in MG : \Theta_\mu|_\mathcal{V} = 0\}$,

(2) $\mathcal{V}_Q = \{T \in \mathcal{B}(H) : \Theta_\mu(T) = 0, \quad \forall \mu \in Q\}$,

are bijective and inverse of each other, see diagram below.

$$\begin{tikzcd}
\left\{\begin{matrix} \mathcal{V}\subset\mathcal{B}(H): \\ \mathcal{V},\ \textit{invariant}\end{matrix}\right\} \arrow[rr, bend right, "Q_\mathcal{V}"'] & & \left\{\begin{matrix} Q\subset MG:(MG)\,Q\subset Q, \\ \sigma(C_bG)\text{-}\textit{closed}\end{matrix}\right\} \arrow[ll, bend right, "\mathcal{V}_Q"']
\end{tikzcd}$$

Let us denote by V_Q^Δ the set

$$\mathcal{V}_Q^\Delta = \{T \in \mathcal{B}(H)^\Delta : \Theta_\mu(T) = 0, \quad \forall \mu \in Q\}.$$

The proof of Theorem 5.3.2 requires the following two lemmas.

Lemma 5.3.3. *If $Q \subset MG$ is a $\sigma(C_bG)$-closed left ideal, then*

$${}_{\mathcal{R}G}\langle \mathcal{V}_Q^\Delta \rangle_{\mathcal{R}G} = \mathcal{V}_Q.$$

Proof. After identifying $\mathcal{B}(L_2G)$ with $L_\infty(G) \rtimes_r G$ again, we have that $\Theta_\mu = T_\mu \rtimes \mathrm{Id}$, where T_μ is the left convolution operator associated to μ. We have that

$$\mathcal{V}_Q \;=\; \bigcap_{\mu\in Q} \ker(T_\mu \rtimes \mathrm{Id}), \quad \mathcal{V}_Q^\Delta \;=\; \bigcap_{\mu \in Q} \ker(T_\mu).$$

But know we use that if T is a r-equivariant operator then

$$\begin{aligned}
\ker(T \rtimes \mathrm{Id}) &= \overline{\mathrm{span}}^{\mathrm{w}^*}(\ker(T)\,\mathcal{R}G) \\
&= \overline{\mathrm{span}}^{\mathrm{w}^*}\{f\,\rho_g : f \in \ker(T),\ g \in G\} \\
&\subset {}_{\mathcal{R}G}\langle \ker(T)\rangle_{\mathcal{R}G}\,.
\end{aligned}$$

Using that the intersection of closures is larger that the closure of the intersections we get that

$$\begin{aligned}\mathcal{V}_Q &= \bigcap_{\mu\in Q} \ker(T_\mu \rtimes \mathrm{Id}) \\ &\subset \bigcap_{\mu\in Q} {}_{\mathcal{R}G}\langle \ker(T_\mu)\rangle_{\mathcal{R}G} \\ &\subset {}_{\mathcal{R}G}\Big\langle \bigcap_{\mu\in Q} \ker(T_\mu)\Big\rangle_{\mathcal{R}G} = {}_{\mathcal{R}G}\langle \mathcal{V}_Q^\Delta\rangle_{\mathcal{R}G}.\end{aligned}$$

The other inclusion is trivial since $\mathcal{V}_Q^\Delta \subset \mathcal{V}_Q$ and $\mathcal{V}_Q$ is a $\mathcal{R}G$-bimodule. $\square$

Lemma 5.3.4. *Let $Q \mapsto \mathcal{V}_Q^\Delta$ and $\mathcal{V}^\Delta \mapsto Q_{\mathcal{V}^\Delta}$ be as above, we have that*

(1) $Q_{\mathcal{V}_Q^\Delta} = Q$.

(2) ${\mathcal{V}^\Delta}_{Q_{\mathcal{V}^\Delta}} = \mathcal{V}^\Delta$.

Proof. Let us start by (1). It is trivial that $Q \subset Q_{\mathcal{V}_Q}$. We only have to prove the reverse inclusion. Assume that $Q_{\mathcal{V}_Q}$ is greater that Q. Then by the Hanh-Banach Theorem, for any $\mu_0 \in Q_{\mathcal{V}_Q} - Q$ we can take a functional $f_0 \in C_b(G)$ such that $\langle \mu_0, f_0\rangle \neq 0$ but $\langle \mu, f_0\rangle = 0$, for every $\mu \in Q$. Since Q is a translation invariant space we have that $\mu * f_0 = 0$ for every $\mu \in Q$ but $\mu_0 * f_0 \neq 0$. The first condition implies that $f_0 \in \mathcal{V}_Q$, which contradict the fact that $\mu_0 \in Q_{\mathcal{V}_Q}$.

For (2) it is again clear that $\mathcal{V}^\Delta \subset \mathcal{V}_{Q_\mathcal{V}}$ and we only have to prove the converse inclusion. By similar means using the Hanh-Banach theorem and the translation invariance of V^Δ we get the result. $\square$

Now, we can proceed to prove the main correspondence theorem.

Proof (of Theorem 5.3.2). Let us start seeing that $Q_{\mathcal{V}_Q}$ is a $\sigma(C_bG)$-closed ideal. Notice that, $\mu * f = 0$ if and only if $\langle g, \mu * f\rangle = 0$ for every $g \in L_1(G)$, but $\langle g, \mu * f\rangle = \langle \mu, \tilde{f} * g\rangle$, where $\tilde{f}(x) = f(x^{-1})$. Since $\tilde{f} * g$ is a right uniformly bounded function in $C_b(G)$, the kernel of $\mu \mapsto \mu * f$ is $\sigma(C_bG)$-closed and so is $Q_{\mathcal{V}_Q}$. The fact that $\mathcal{V}_Q$ is a weak-$*$ closed $\mathcal{R}G$-bimodule is immediate since Θ_μ is weak-$*$ continuous $\mathcal{R}G$-bimodular map. The fact that is Δ-invariant follows from 5.3.3. To prove that $Q = Q_{\mathcal{V}_Q}$ we just apply the following lemmas.

$$\begin{aligned}Q &= Q_{\mathcal{V}_Q^\Delta} && \text{(by Lemma 5.3.4)}\\ &= Q_{{}_{\mathcal{R}G}\langle \mathcal{V}_Q^\Delta\rangle_{\mathcal{R}G}} \\ &= Q_{\mathcal{V}_Q} && \text{(by Lemma 5.3.3).}\end{aligned}$$

Similarly, taking the $\mathcal{R}G$-bimodules generated by the left and the right hand side of (2), we get that

$${}_{\mathcal{R}G}\langle \mathcal{V}^\Delta\rangle_{\mathcal{R}G} = {}_{\mathcal{R}G}\langle \mathcal{V}^\Delta_{Q_{\mathcal{V}^\Delta}}\rangle_{\mathcal{R}G} = \mathcal{V}_{Q_{\mathcal{V}^\Delta}} = \mathcal{V}_{Q_{{}_{\mathcal{R}G}\langle \mathcal{V}^\Delta\rangle_{\mathcal{R}G}}}.$$

But, by Δ-invariance, the leftmost element is $\mathcal{V}$ and the rightmost is $\mathcal{V}_{Q_\mathcal{V}}$ and we conclude. $\square$

Remark 5.3.5. We have exposed here the theory of invariant quantum relations for $\mathcal{L}G$. The same proof above works, after [NRS08] and [JNR09], for a general quantum group $(\mathcal{A}, \Delta)$ just by replacing left ideals in MG by left ideals in $M_{\mathrm{cb}}A(\mathcal{A})$.

Recall that if $G = \mathbb{Z}^n$, or any other abelian discrete group, then $\mathcal{L}G = L_\infty(\mathbb{T}^n)$ and any ideal Q of $MG = \ell_1(\mathbb{Z}^n)$ correspond to a closed subset $C_Q \subset \hat{G}$ and such correspondence in injective. Nevertheless, not every ideal in MG is $\sigma(C_bG)$-closed and therefore not all closed subsets will appear in the image of the correspondence. We have that, in the invariant case, any quantum relation $\mathcal{V}$ over $L_\infty(\mathbb{T}^n)$ is actually a topological relation given by

$$(\theta_1, \theta_2) \in R \iff \theta_1^{-1}\,\theta_2 \in C,$$

where $C \subset \mathbb{T}^n$ is a closed set.

5.4 Remarks on L_p-L_q versions of Quantum Relations

In the introduction of [Wea12] it is stated that the natural, albeit naive, candidate for quantized relations over a von Neumann algebra $\mathcal{M}$ are the projections on $\mathcal{M} \,\overline{\otimes}\, \mathcal{M}_{\mathrm{op}}$, but that such object do not have desirable properties. The question of which properties are missed is left unanswered there. Our aim here is to give an intuitive explanation on why there is no well-behaved composition operation between projection in $\mathcal{M} \,\overline{\otimes}\, \mathcal{M}_{\mathrm{op}}$. After that, we will see that there is a larger family of *generalized quantum relations* that contains both quantum relations and projections in $\mathcal{M} \,\overline{\otimes}\, \mathcal{M}_{\mathrm{op}}$ as particular cases.

Recall from Chapter 1 that if $\mathcal{M}$ is a von Neumann algebra and ϕ is a normal, faithful and semifinite weight we can define the noncommutative L_p-spaces $L_p(\mathcal{M}, \phi)$, or simply $L_p(\mathcal{M})$ if ϕ is understood from the context, see [PX03]. Apart from being compatible with interpolation, such spaces satisfy that $L_p(\mathcal{M})^* = L_{p'}(\mathcal{M}_{\mathrm{op}})$. It is also known that $\mathcal{CB}(L_2(\mathcal{M})) = \mathcal{B}(L_2(\mathcal{M}))$. The spaces $L_p(\mathcal{M})$ can be turned into $\mathcal{M}$-bimodules. Indeed, we have two commuting c.b. representations $l_p : \mathcal{M} \to \mathcal{CB}(L_p(\mathcal{M}))$ and $r_p : \mathcal{M}_{\mathrm{op}} \to \mathcal{CB}(L_p(\mathcal{M}))$ generalizing the commuting actions in the GNS construction of $\mathcal{M}$ when $p = 2$. The module structure of noncommtutative-L_p has been studied in [JS05]. Let us denote by $S' \subset \mathcal{CB}(L_p(\mathcal{M}))$ the commutant of $S \subset \mathcal{CB}(L_p(\mathcal{M}))$, by [JS05, Theorem 1.5], we have that

$$\begin{aligned} l_p[\mathcal{M}]' &= r_p[\mathcal{M}_{\mathrm{op}}] \\ r_p[\mathcal{M}_{\mathrm{op}}]' &= l_p[\mathcal{M}]. \end{aligned}$$

Let us denote by $\mathcal{CB}_{p,q}$ the operator space given by $\mathcal{CB}(L_p(\mathcal{M}), L_q(\mathcal{M}))$. Such spaces have a natural predual given by

$$\begin{aligned} \mathcal{CB}(L_p(\mathcal{M}), L_q(\mathcal{M})) &= \mathcal{CB}(L_p(\mathcal{M}), L_{q'}(\mathcal{M}_{\mathrm{op}})^*) \\ &= \mathcal{CB}(L_p(\mathcal{M}), \mathbb{C}) \otimes_{\mathcal{F}} L_{q'}(\mathcal{M}_{\mathrm{op}})^* \quad \text{(by [Pis03, Th. 4.1])} \\ &= (L_p(\mathcal{M}) \,\widehat{\otimes}\, L_{q'}(\mathcal{M}_{\mathrm{op}}))^*. \end{aligned}$$

There are natural left actions on $\mathcal{CB}_{p,q}$ by $r_p[\mathcal{M}_{\mathrm{op}}]$ and $l_p[\mathcal{M}]$ and right actions by $r_q[\mathcal{M}_{\mathrm{op}}]$, $l_q[\mathcal{M}]$. We say that a subspace $\mathcal{V} \subset \mathcal{CB}_{p,q}$ is a (p,q)*-quantum relation* over $\mathcal{M}$ iff $\mathcal{V}$ is weak-$*$ closed, with respect to the predual $L_p(\mathcal{M}) \,\widehat{\otimes}\, L_{q'}(\mathcal{M}_{\mathrm{op}})$, and a $r_p[\mathcal{M}_{\mathrm{op}}]$-$r_q[\mathcal{M}_{\mathrm{op}}]$-bimodule. It is easily shown that such relations are independent of ϕ. We also have the following.

Proposition 5.4.1.

(i) *Quantum relations over $\mathcal{M}$ correspond to $(2,2)$-quantum relations.*

(ii) *Projections in $\mathcal{M} \,\overline{\otimes}\, \mathcal{M}_{\mathrm{op}}$ are in bijective correspondence with $(1,\infty)$-quantum relations.*

Proof. The proof of (i) is immediate since $\mathcal{CB}_{2,2} = \mathcal{B}(L_2(\mathcal{M}))$, $l_2 : \mathcal{M} \to \mathcal{B}(L_2(\mathcal{M}))$ is the GNS representation of $\mathcal{M}$ and $r_2[\mathcal{M}]$ is just the commutant M' for that representation. In order to prove (ii) we need to use that the map $j : \mathcal{M} \otimes_{\mathrm{alg}} \mathcal{M} \to \mathcal{CB}(\mathcal{M}_*, \mathcal{M})$ given by linear extension of

$$j(x \otimes y)(\xi) = \langle y, \xi \rangle \, x$$

extends to a weak-$*$ continuous complete isomorphism $j : \mathcal{M} \overline{\otimes} \mathcal{M} \to \mathcal{CB}(\mathcal{M}_*, \mathcal{M})$, see [Pis03, Theorem 2.5.2]. we have that

$$\mathcal{CB}_{1,\infty} = \mathcal{CB}(L_1(\mathcal{M}), \mathcal{M}) = \mathcal{CB}(\mathcal{M}_*^{\mathrm{op}}, \mathcal{M}) = \mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}.$$

But, if $T = j(x \otimes y)$, we have that $r_\infty(z) \, T \, r_1(t) = j(z\, x \otimes y\, t) = j((z \otimes t)\,(x \otimes y))$ and so a subspace $\mathcal{V} \subset \mathcal{CB}_{1,\infty}$ is $r_\infty[\mathcal{M}_{\mathrm{op}}]$-$r_1[\mathcal{M}_{\mathrm{op}}]$-bimodular iff, after seeing $\mathcal{V}$ as a subspace of $\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}$, it is a left ideal. Since the map j is an isomorphism for the weak-$*$ topology, $\mathcal{V} \subset \mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}$ is also weak-$*$ closed. But any weak-$*$ closed left ideal is of the form $\mathcal{V} = (\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}) \, P$, where $P \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}})$. $\square$

Remark 5.4.2. The result above explains intuitively why we cannot expect to define a well-behaved composition operation between projections $P, Q \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}})$. That composition will be carried to the composition of operators in $\mathcal{CB}(L_1(\mathcal{M}), \mathcal{M})$ but that cannot be done, in general, since $\mathcal{M}$ does not embeds canonically in $L_1(\mathcal{M})$.

It is natural to ask whether (p,q)-quantum relations are in correspondence with left ideals $J \subset \mathcal{CB}^{\sigma}_{r_q[\mathcal{M}_{\mathrm{op}}]\text{-}r_p[\mathcal{M}_{\mathrm{op}}]}(\mathcal{CB}_{p,q})$ suitably closed in some weak topology. The following proposition asserts that this is the case when $(p,q) = (1, \infty)$.

Proposition 5.4.3. *The map $\Phi^{1,\infty}(x \otimes y) = l_\infty(x)\, l_1(y)$ extends to a weakly continuous complete isomorphism*

$$\Phi^{1,\infty} : \mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}} \to \mathcal{CB}^{\sigma}_{r_\infty[\mathcal{M}_{\mathrm{op}}]\text{-}r_1[\mathcal{M}_{\mathrm{op}}]}(\mathcal{CB}_{1,\infty}).$$

Under such correspondence any weakly closed left ideal J is of the form $J = P\,(\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}})$ and its associated bimodule $\mathcal{V}_J$ corresponds, under the bijection in (ii)*, to $P^\perp \in \mathcal{P}(\mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}})$.*

To prove the theorem above just notice that if $\mathcal{N}$ is a von Neumann algebra, normal right $\mathcal{N}$-modular maps $T : \mathcal{N} \to \mathcal{N}$ are given by left multiplication. Then, by applying that result to $\mathcal{N} = \mathcal{M} \overline{\otimes} \mathcal{M}_{\mathrm{op}}$ and using Proposition 5.4.1 we conclude.

The discussion above leaves two natural open problems.

Problem 5.4.4.

- **(P1.)** Determine whether the double annihilator relation in Theorem 5.2.3 between modules and ideals holds in general for (p,q)-quantum relations.
- **(P2.)** define an operator space tensor product $\otimes_{p,q}$ such that the map $\Phi^{p,q} : \mathcal{M} \otimes_{\mathrm{alg}} \mathcal{M}_{\mathrm{op}} \to \mathcal{CB}^{\sigma}_{r_q[\mathcal{M}_{\mathrm{op}}]\text{-}r_p[\mathcal{M}_{\mathrm{op}}]}(\mathcal{CB}_{p,q})$ extends as a complete isometry to $\mathcal{M} \otimes_{p,q} \mathcal{M}_{\mathrm{op}}$.

5.5 W^*-metrics and c.b. Gaussian bounds

The aim of this section is to explain the original motivation that guided us into studying quantum relations. Such motivation was the necessity on [GPJP15] and [GPJP16] of expressing *off-diagonal* bounds in the context of noncommutative metric spaces. Recall that in the classical case an operator $T = [a_{x,y}]_{x,y\in X}$ affiliated to $\mathcal{B}(\ell_2 X)$ has off-diagonal bounds if certain norms of

$$\big[a_{x\,y}\,\chi_{\{(x,y):d(x,y)>r\}}\big]_{x,y\in X}$$

decay in terms of $r > 0$. Earlier definition of *quantum metric spaces* in the C^*-algebraic framework, see [Rie04b], [Rie04a], do not provide a natural way of formulating such notion. On the other hand the notion of W^*-metric introduced by Kuperberg and Weaver in [KW12] seems particularly well suited to the task since a W^*-metric is a noncommutative generalization of the bundle of band matrices of width $r > 0$.

The upbringing of the notion of quantum relation is tightly connected with the concept of W^*-metric space introduced by Kuperberg and Weaver in [KW12]. Let us recall briefly such definition.

Definition 5.5.1 ([KW12, Definition 2.1(a)/2.3]). A family of subspaces $\mathbb{V} = (\mathcal{V}_r)_{r\geq 0}$ of $\mathcal{B}(H)$ is a W^**-pseudometric* over $\mathcal{M} \subset \mathcal{B}(H)$ iff

(i) Each $\mathcal{V}_r$ is a quantum relation over $\mathcal{M}$.

(ii) Each $\mathcal{V}_r$ is symmetric, i.e. $\mathcal{V}_r^* = \mathcal{V}_r$.

(iii) $\mathcal{V}_r \cdot \mathcal{V}_s \subset \mathcal{V}_{r+s}$.

(iv) $\displaystyle\bigcap_{s>t} \mathcal{V}_s = \mathcal{V}_t$.

We say that $\mathbb{V}$ is a W^**-metric* iff $\mathcal{V}_0 = \mathcal{M}$.

Notice that, if $\mathcal{M} = \ell_\infty(X)$ is a discrete measure space, then every $\mathcal{V}_r$ corresponds to a relation $R_r \subset X\times X$. Condition (ii) becomes usual symmetry for R_r. Defining a function $d_{\mathbb{V}}(x,y) = \inf\{r : (x,y) \in R_r\}$ gives that (iv) is the triangular inequality and so $d_{\mathbb{V}}$ is a classical (pseudo)metric.

Classically, a metric measure space is a triple (X, μ, d) where μ is a measure and d is a metric such that the Borel σ-algebra generated by d is composed of measurable sets. The noncommutative version of a measure space is generally regarded as a pair $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a von Neumann algebra and $\tau : \mathcal{M}_+ \to [0,\infty]$ normal, semifinite and faithful trace (or more generally a weight). Using W^*-metrics in this context gives a good noncommutative generalization of metric measure spaces. There are other, earlier, notions of quantized metric spaces, see for instance [Rie04b], but W^*-metrics have some advantages. One of them is that they provide a more natural framework for studying both finite speed of propagation and *off-diagonal* bounds associated with a Markovian semigroup over $(\mathcal{M}, \tau)$. Recall that, in the classical example of the heat semigroup on $\mathbb{R}^n$, $S_t = e^{-t\,(-\Delta)}$ and its kernel k_t satisfies Gaussian bounds of the form

$$\big\| k_t(x,y)\,\chi_{\{(x,y):d(x,y)>r\}} \big\|_{L_\infty(\mathbb{R}^n\times\mathbb{R}^n)} \lesssim_{(n)} \frac{e^{-\frac{r^2}{4\,t}}}{\sqrt{t^n}}. \tag{5.5.1}$$

Notice that, if $J_{\mathcal{V}_r}$ is generated by a projection $P_r \in \mathcal{P}(\mathcal{M}\,\overline{\otimes}\,\mathcal{M}_op)$ and the semigroup S_t can be expressed as an integral operator by

$$S_t(x) = \tau\big\{k_t\,(\mathbf{1}\otimes x)\big\},$$

for some k_t affiliated to $\mathcal{M}\,\overline{\otimes}\,\mathcal{M}_{\mathrm{op}}$, then the off-diagonal restriction is just $k_t\,P_r$ and we can generalize (5.5.1) by bounding such element. Since, in general, ideals in $\mathcal{M}\otimes_{eh}\mathcal{M}$ are not principal, such projection doesn't exist. Nevertheless, we can take $\Phi_s(S_t)$ for $s \in \mathrm{Ball}(J_{\mathcal{V}_r})$, the unit ball of $J_{\mathcal{V}_r}$, obtaining noncommutative Gaussian bounds of the form

$$\sup_{s \in \mathrm{Ball}(J_{\mathcal{V}_r})} \big\| \Phi_s(S_t) \big\|_{\mathcal{CB}(L_1(\mathcal{M}),\mathcal{M})} \lesssim \frac{e^{-\beta \frac{r^2}{t}}}{\sqrt{t^n}}.$$

As we hinted at the end of Chapter 3, another Harmonic analysis concept that seems natural to formulate in the context of W^*-metrics is finite speed of propagation for the wave equation. If $S_t = e^{-t\,(-\Delta)}$ is the heat equation in $\mathbb{R}^n$, then its associated wave equation is given by

$$\partial_{t\,t}^2 f_t + (-\Delta) f_t = 0.$$

The solution of such equation have *finite speed of propagation*, meaning that if f_t is a solution of the above equation and $\mathrm{supp}[f_0] = K$, after time $t > 0$ the support of f_t is contained in

$$B_t(K) = \bigcup_{x \in K} B_t(x).$$

Such condition can be defined trivially using W^*-metrics as follows.

Definition 5.5.2. We say that a Markovian semigroup over $S_t = e^{tA}$ have finite speed of propagation (with respect to some W^*-metric $\mathbb{V}$) iff

$$\cos(t\sqrt{A}) \in \mathcal{V}_t, \quad \forall\, t > 0.$$

Observe that the definition makes perfect sense since, without loss of generality, we can assume $\mathcal{V} \subset \mathcal{B}(L_2(\mathcal{M},\tau))$ and clearly $\cos(t\sqrt{A})$ is bounded in L_2. The intuition behind is that $x_t = \cos(t\sqrt{A})\, x$ satisfies the equation $\partial_{t\,t}^2 x_t + A x_t = 0$ with $x_0 = x$.

Gaussian bounds and finite speed of propagation are equivalent after assuming certain hypothesis, see [Sik96] [Sik04]. Generalizing such results and exploring the connections with locality in the noncommutative setting is the goal of a forthcoming article.

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Bibliography

[AEG94] Scot Adams, George A. Elliott, and Thierry Giordano. Amenable actions of groups. *Trans. Amer. Math. Soc.*, 344(2):803–822, 1994.

[Ale94] G. Alexopoulos. Spectral multipliers on Lie groups of polynomial growth. *Proc. Amer. Math. Soc.*, 120(3):973–979, 1994.

[Ale01] G. Alexopoulos. Spectral multipliers on discrete groups. *Bulletin of the London Mathematical Society*, 33:417–424, 7 2001.

[BdlHV08] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's Property (T)*. New Mathematical Monographs. Cambridge University Press, 2008.

[Ben14] J. Bennett. Optimal control of singular Fourier multipliers by maximal operators. *Anal. & PDE*, 7(6):1317–1338, 2014.

[BF91] Marek Bożejko and Gero Fendler. Herz-schur multipliers and uniformly bounded representations of discrete groups. *Archiv der Mathematik*, 57(3):290–298, 1991.

[BF06] M. Bożejko and G. Fendler. A note on certain partial sum operators. In *Quantum probability*, volume 73 of *Banach Center Publ.*, pages 117–125. Polish Acad. Sci. Inst. Math., Warsaw, 2006.

[BL12] J. Bergh and J. Lofstrom. *Interpolation Spaces: An Introduction*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.

[BM04] D.P. Blecher and C.L. Merdy. *Operator Algebras and Their Modules: An Operator Space Approach*. London Mathematical Society monographs. Clarendon, 2004.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[BS92a] D. P. Blecher and R. R. Smith. The dual of the Haagerup tensor product. *J. London Math. Soc. (2)*, 45(1):126–144, 1992.

[BS92b] D. P. Blecher and R. R. Smith. The dual of the Haagerup tensor product. *J. London Math. Soc. (2)*, 45(1):126–144, 1992.

[CCJ$^+$01] P.A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability.

[CdlS15] M. Caspers and M. de la Salle. Schur and Fourier multipliers of an amenable group acting on non-commutative L_p-spaces. *Trans. Amer. Math. Soc.*, 2015.

[CH89] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.

[Chr91] Michael Christ. L_p-bounds for spectral multipliers on nilpotent groups. *Transactions of the American Mathematical Society*, 328(1):73–81, 1991.

[Con75] A. Connes. A factor not anti-isomorphic to itself. *Ann. Math.*, 101:536–554, 1975.

[Cou93] Thierry Coulhon. Iteration de moser et estimation gaussienne du noyau de la chaleur. *J. Operator Theory*, 29(1):157–165, 1993.

[CPPR15] Martijn Caspers, Javier Parcet, Mathilde Perrin, and Éric Ricard. Noncommutative de leeuw theorems. In *Forum of Mathematics, Sigma*, volume 3, page e21. Cambridge Univ Press, 2015.

[Dav90] E.B. Davies. *Heat Kernels and Spectral Theory*. Cambridge Tracts in Mathematics. Cambridge University Press, 1990.

[Dav95] E.B. Davies. Uniformly elliptic operators with measurable coefficients. *J. Funct. Anal.*, 132(1):141–169, 1995.

[dlH88] Pierre de la Harpe. Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(14):771–774, 1988.

[dlSM14] M. de la Salle and T. Mei. Complete boundedness of the Heat semigroups on the von Neumann Algebra of hyperbolic groups. *Trans. Amer. Math. Soc.*, 2014.

[DOS02] T.X. Duong, E.M. Ouhabaz, and A. Sikora. Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.*, 196(2):443–485, 2002.

[DP89] E. B. DAVIES and M. M. H. PANG. Sharp heat kernel bounds for some laplace operators. *The Quarterly Journal of Mathematics*, 40(3):281–290, 1989.

[Duo01] Javier Duoandikoetxea. *Fourier analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[EK87] E. G. Effros and A. Kishimoto. Module maps and Hochschild-Johnson cohomology. *Indiana Univ. Math. J.*, 36(2):257–276, 1987.

[EKR93] E. G. Effros, J. Kraus, and Z.-J. Ruan. On two quantized tensor products. In *Operator algebras, mathematical physics, and low-dimensional topology (Istanbul, 1991)*, volume 5 of *Res. Notes Math.*, pages 125–145. A K Peters, Wellesley, MA, 1993.

[ER88] E. G. Effros and Z.-J. Ruan. On matricially normed spaces. *Pacific J. Math.*, 132(2):243–264, 1988.

[ER91] E. G. Effros and Z.-J. Ruan. Self-duality for the haagerup tensor product and hilbert space factorizations. *Journal of Functional Analysis*, 100(2):257 – 284, 1991.

[ER00] E. G. Effros and Z.-J. Ruan. *Operator Spaces*, volume 23 of *London Mathematical*

Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 2000.

[ER03] E. G. Effros and Z.-J. Ruan. Operator space tensor products and Hopf convolution algebras. *J. Operator Theory*, 50(1):131–156, 2003.

[Erd86] J. A. Erdos. Reflexivity for subspace maps and linear spaces of operators. *Proc. London Math. Soc. (3)*, 52(3):582–600, 1986.

[Fol95] G.B. Folland. *A Course in Abstract Harmonic Analysis.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[FOT10] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes.* De Gruyter Studies in Mathematics. De Gruyter, 2010.

[FS82] G.B. Folland and E.M. Stein. *Hardy Spaces on Homogeneous Groups.* Mathematical Notes - Princeton University Press. Princeton University Press, 1982.

[Fur73] Harry Furstenberg. Boundary theory and stochastic processes on homogeneous spaces. In *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, pages 193–229. Amer. Math. Soc., Providence, R.I., 1973.

[Goo14] R.W. Goodman. *Nilpotent Lie Groups.* Springer, 2014.

[GP16a] A. M. González Pérez. Crossed-products extensions of L_p-bounds for amenable actions. *preprint*, 2016.

[GP16b] A. M. González Pérez. Gaussian bounds for Markov semigroups ove a von Neumann algebra. *(work in progress)*, 2016.

[GPJP15] A. M. González Pérez, M. Junge, and J. Parcet. Smooth Fourier multipliers in group von Neumann algebras via Sobolev dimension. *http://arxiv.org/abs/1505.05316*, 2015.

[GPJP16] A. M. González Pérez, M. Junge, and J. Parcet. Singular integral operators for quantum Euclidean spaces. *preprint*, 2016.

[Haa78] U. Haagerup. An example of a non nuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978.

[Haa79a] U. Haagerup. Operator-valued weights in von Neumann algebras. I. *J. Funct. Anal.*, 32(2):175–206, 1979.

[Haa79b] U. Haagerup. Operator-valued weights in von Neumann algebras. II. *J. Funct. Anal.*, 33(3):339–361, 1979.

[Haa85] Uffe Haagerup. Injectivity and decomposition of completely bounded maps. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 170–222. Springer, Berlin, 1985.

[Haa86] U. Haagerup. Decomposition of completely bounded maps on operator algebras. *(unpublished manuscript)*, 1986.

[Haa79] U. Haagerup. An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50(3):279–293, 1978/79.

[Har99] A. Harcharras. Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets. *Studia Math.*, 137(3):203–260, 1999.

[Heb95] W. Hebisch. Functional calculus for slowly decaying kernels. *preprint*, 1995.

[Her72] C. Herz. Problems of extrapolation and spectral synthesis on groups. In *Conference on Harmonic Analysis (Univ. Maryland, College Park, Md., 1971)*, pages 157–166. Lecture Notes in Math., Vol. 266. Springer, Berlin, 1972.

[HR00] Nigel Higson and John Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000.

[HS] Andrzej Hulanicki and Elias M Stein. Marcinkiewicz multiplier theorem for stratified groups.

[JLMX06] M. Junge, C. Le Merdy, and Q. Xu. H^∞ functional calculus and square functions on noncommutative L^p-spaces. *Astérisque*, (305), 2006.

[JM10] M. Junge and T. Mei. Noncommutative Riesz transforms—A probabilistic approach. *Amer. J. Math.*, 132(3):611–680, 2010.

[JM12] M. Junge and T. Mei. BMO spaces associated with semigroups of operators. *Math. Ann.*, 352(3):691–743, 2012.

[JMP14a] M. Junge, T. Mei, and J. Parcet. Smooth Fourier multipliers on group von Neumann algebras. *Geom. Funct. Anal.*, 24(6):1913–1980, 2014.

[JMP14b] Marius Junge, Tao Mei, and Javier Parcet. Algebraic Calderón-Zygmund theory. *(work in progress)*, 2014.

[JMP14c] Marius Junge, Tao Mei, and Javier Parcet. Noncommutative riesz transforms–dimension free bounds and fourier multipliers. *Journal European Mathematical Society (to appear)*, 2014.

[JMP15] M. Junge, T. Mei, and J. Parcet. Noncommutative Riesz transforms—Dimension free bounds and Fourier multipliers. ArXiv:1407.2475, 2015.

[JNR09] M. Junge, M. Neufang, and Z.-J. Ruan. A representation theorem for locally compact quantum groups. *Internat. J. Math.*, 20(3):377–400, 2009.

[Jol90] Paul Jolissaint. Rapidly decreasing functions in reduced C^*-algebras of groups. *Trans. Amer. Math. Soc.*, 317(1):167–196, 1990.

[JS05] M. Junge and D. Sherman. Noncommutative L^p modules. *J. Operator Theory*, 53(1):3–34, 2005.

[Jun02] M. Junge. Doob's inequality for non-commutative martingales. *J. Reine Angew. Math.*, 549:149–190, 2002.

[JX03] M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. *Ann.*

Probab., 31(2):948–995, 2003.

[JX07] M. Junge and Q. Xu. Noncommutative maximal ergodic theorems. *J. Amer. Math. Soc.*, 20(2):385–439, 2007.

[KR97] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras: Elementary theory*. Fundamentals of the Theory of Operator Algebras. American Mathematical Society, 1997.

[KW99] Eberhard Kirchberg and Simon Wassermann. Exact groups and continuous bundles of C^*-algebras. *Math. Ann.*, 315(2):169–203, 1999.

[KW12] G. Kuperberg and N. Weaver. A von Neumann algebra approach to quantum metrics. *Mem. Amer. Math. Soc.*, 215(1010):v, 1–80, 2012.

[Lan95] E.C. Lance. *Hilbert C^*-Modules: A Toolkit for Operator Algebraists*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1995.

[Lar82] D. R. Larson. Annihilators of operator algebras. In *Invariant subspaces and other topics (Timişoara/Herculane, 1981)*, volume 6 of *Operator Theory: Adv. Appl.*, pages 119–130. Birkhäuser, Basel-Boston, Mass., 1982.

[LdlS11] V. Lafforgue and M. de la Salle. Noncommutative L^p-spaces without the completely bounded approximation property. *Duke Math. J.*, 160(1):71–116, 10 2011.

[LP86] F. Lust-Piquard. Inégalités de Khintchine dans C_p $(1 < p < \infty)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(7):289–292, 1986.

[LPP91] F. Lust-Piquard and G. Pisier. Noncommutative Khintchine and Paley inequalities. *Ark. Mat.*, 29(2):241–260, 1991.

[Mey76] P.A. Meyer. L'operateur carré du champ. In P.A. Meyer, editor, *Séminaire de Probabilités X Université de Strasbourg*, volume 511 of *Lecture Notes in Mathematics*, pages 142–161. Springer Berlin Heidelberg, 1976.

[MP09] Tao Mei and Javier Parcet. Pseudo-localization of singular integrals and noncommutative littlewoodpaley inequalities. *International Mathematics Research Notices*, 2009(8):1433–1487, 2009.

[NR11] S. Neuwirth and E. Ricard. Transfer of Fourier multipliers into Schur multipliers and sumsets in discrete group. *Can. J. Math.*, 63(5):1161–1187, 2011.

[NRS08] M. Neufang, Z-J. Ruan, and N. Spronk. Completely isometric representations of $M_{cb}A(G)$ and $\mathrm{UCB}(G)^*$. *Transactions of the American Mathematical Society*, 360(3):1133–1161, 2008.

[Oza00] N. Ozawa. Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8):691–695, 2000.

[Oza06] Narutaka Ozawa. Amenable actions and applications. In *International Congress of Mathematicians. Vol. II*, pages 1563–1580. Eur. Math. Soc., Zürich, 2006.

[Par09] Javier Parcet. Pseudo-localization of singular integrals and noncommutative calderón–

zygmund theory. *Journal of Functional Analysis*, 256(2):509–593, 2009.

[Pat88] Alan L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1988.

[Pau86] V. I. Paulsen. *Completely bounded maps and dilations*, volume 146 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1986.

[Ped79] G. K. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979.

[Pis95a] Gilles Pisier. Multipliers and lacunary sets in non-amenable groups. *Amer. J. Math.*, 117(2):337–376, 1995.

[Pis95b] Gilles Pisier. Regular operators between non-commutative L_p-spaces. *Bull. Sci. Math.*, 119(2):95–118, 1995.

[Pis98] G. Pisier. Non-commutative vector valued L_p-spaces and completely p-summing maps. *Astérisque*, (247), 1998.

[Pis03] G. Pisier. *Introduction to Operator Space Theory*, volume 294 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2003.

[PR16] Javier Parcet and Keith M Rogers. Twisted hilbert transforms vs kakeya sets of directions. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2016(710):137–172, 2016.

[PX03] G. Pisier and Q. Xu. Non-commutative L^p-spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1459–1517. North-Holland, Amsterdam, 2003.

[Rie02] M. A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651 (electronic), 2002.

[Rie04a] M. A. Rieffel. Compact quantum metric spaces. In *Operator algebras, quantization, and noncommutative geometry*, volume 365 of *Contemp. Math.*, pages 315–330. Amer. Math. Soc., Providence, RI, 2004.

[Rie04b] M. A. Rieffel. Gromov-hausdorff distance for quantum metric spaces. *Mem. Amer. Math. Soc*, pages 1–65, 2004.

[Rud90] Walter Rudin. *Fourier analysis on groups.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.

[Rya13] R.A. Ryan. *Introduction to Tensor Products of Banach Spaces.* Springer Monographs in Mathematics. Springer London, 2013.

[SC02] Laurent Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2002.

[SC09] L. Saloff-Coste. Sobolev inequalities in familiar and unfamiliar settings. In *Sobolev spaces in mathematics. I*, volume 8 of *Int. Math. Ser. (N. Y.)*, pages 299–343. Springer, New York, 2009.

[Sik96] A. Sikora. Sharp pointwise estimates on heat kernels. *Quart. J. Math. Oxford Ser. (2)*, 47(187):371–382, 1996.

[Sik04] A. Sikora. Riesz transform, Gaussian bounds and the method of wave equation. *Math. Z.*, 247(3):643–662, 2004.

[Smi91] R. R. Smith. Completely bounded module maps and the Haagerup tensor product. *Journal of Functional Analysis*, 102(1):156 – 175, 1991.

[Spr04] N. Spronk. Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras. *Proc. London Math. Soc. (3)*, 89(1):161–192, 2004.

[Ste70] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[Sti55] W. Forrest Stinespring. Positive functions on c¡sup¿*¡/sup¿-algebras. *Proceedings of the American Mathematical Society*, 6(2):211–216, 1955.

[Tak73] Masamichi Takesaki. Duality in crossed products and von Neumann algebras of type III. *Bull. Amer. Math. Soc.*, 79:1004–1005, 1973.

[Tak79] M. Takesaki. *Theory of operator algebras. I.* Springer-Verlag, New York-Heidelberg, 1979.

[Tak03] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.

[Var85a] V. S. Varadarajan. *Geometry of quantum theory.* Springer-Verlag, New York, second edition, 1985.

[Var85b] N. Th. Varopoulos. Hardy-Littlewood theory for semigroups. *J. Funct. Anal.*, 63(2):240–260, 1985.

[VD14] A. Van Daele. Locally compact quantum groups. A von Neumann algebra approach. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 10:Paper 082, 41, 2014.

[VSCC92] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1992.

[Wea12] N. Weaver. Quantum relations. *Mem. Amer. Math. Soc.*, 215(1010):v–vi, 81–140, 2012.

[Wen52] J. G. Wendel. Left centralizers and isomorphisms of group algebras. *Pacific J. Math.*, 2:251–261, 1952.

[Wit81] Gerd Wittstock. Ein operatorwertiger hahn-banach satz. *Journal of Functional Analysis*, 40(2):127 – 150, 1981.

[Wol07] J.A. Wolf. *Harmonic Analysis on Commutative Spaces.* Mathematical surveys and monographs. American Mathematical Society, 2007.

[Zim77] Robert J. Zimmer. Hyperfinite factors and amenable ergodic actions. *Invent. Math.*, 41(1):23–31, 1977.

[Zim78a] Robert J. Zimmer. Amenable ergodic group actions and an application to Poisson boundaries of random walks. *J. Functional Analysis*, 27(3):350–372, 1978.

[Zim78b] Robert J. Zimmer. Amenable pairs of groups and ergodic actions and the associated von Neumann algebras. *Trans. Amer. Math. Soc.*, 243:271–286, 1978.

[Zim78c] Robert J. Zimmer. Induced and amenable ergodic actions of Lie groups. *Ann. Sci. École Norm. Sup. (4)*, 11(3):407–428, 1978.

[Zim84] R. J. Zimmer. *Ergodic theory and semisimple groups*, volume 81 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.